

Decaphonic piano

Its mathematics, physics and sound

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GRIEG meets Chopin
Warszawa, July 11, 2023

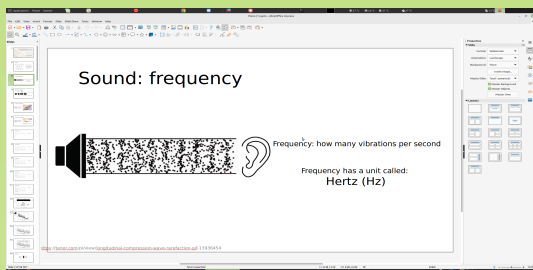
- I am a mathematician from the Center for Theoretical Physics of PAS, and a coordinator of the grant **SCREAM** (Symetry, Curvature Reductions, and EquivAlence Methods) from the **National Science Center of Poland**, in the **GRIEG** grant scheme, founded from the **Norwegian Financial Mechanism 2014-2021** (with project registration number 2019/34/H/ST1/00636.)
- Related to this grant, there is a conference **GRIEG meets Chopin** being held now in Warsaw, and I tried to make a concert for the conference participants. I had an idea of employing a **pianist**, who has both **Chopin** and **Grieg** in her repertoire to play the concert, but I failed.

- Then I thought that in **jazz**, one can superpose several compositions in one piece. Through my friends I got a contact to a jazz pianist I admire, namely **Leszek Mozdzer**, and to my surprise he agreed to give a concert for our conference, provided that I will help him to built a piano he always dreamed about. Actually he needed a mathematical help for this built, and has chosen **the first mathematician who ever contacted him**.
- It turned out that the mathematics he needed, was not demanding at all, but then I realised that an actual implementation of the frequencies which Leszek needs in his **piano** suffers from lack of solutions to a number of **physical** and **technological** problems.

- So I went to my Alma Mater, the **Physics Department of University of Warsaw**, and asked for a PhD student who is very talented in both **experimental** physics and **piano music**.
- It is how I encountered **Aleksander Bogucki** without whom this project would never even had been started.
- Aleksander introduced me to **Andrzej Włodarczyk**, a world expert in the **historic piano restoration** - privately Aleksander's longterm **piano tuner**. Mr Włodarczyk joined our team with enthusiasm. He **revealed many of his piano building secrets** for us. This elevated our project to the professional level. Again, without Andrzej Włodarczyk's contribution to the project, we would not be here today.

Part I: MATHEMATICS

What is sound?



- Sound is a **vibration of air pressure that we sense with our ears**¹.
- **The rate** at which these vibrations hit our eardrums is called **the frequency of the sound**. This is measured in **Hertz**, which is the **number of vibrations per second**.

¹Non-decaponic part of this talk uses a lot from the youtube video of **Yuval Nov**. One can consult his video at https://www.youtube.com/watch?v=nK2jYk37Rlg&ab_channel=Formant in case one is lost in this part

Frequency range

Sound: perception

- Typical human hears from
 - 20 Hz
 - 20 kHz = 20 000 Hz

The diagram illustrates the frequency spectrum of sound. It features a horizontal axis with a yellow-to-purple gradient. A black arrow points from left to right, labeled 'Lower frequency' at the start and 'Higher frequency' at the end. Below this, a black wave represents sound. The wave has long, low-frequency cycles on the left (pink background) and short, high-frequency cycles on the right (purple background). A central section (yellow background) is labeled 'Human' and contains a silhouette of a person. Below the human silhouette, a blue bar indicates the 'HUMAN HEARING RANGE' from 20-20,000 Hz, and a red bar indicates the range 95-1300 Hz. At the bottom, three colored boxes represent 'INFRASOUND' (pink), 'HUMAN HEARING RANGE' (yellow), and 'ULTRASOUND' (purple). A URL is provided: <https://bearylabster.com/hearing-range-diagram/>

- The frequency of musical tones is in the range of, say **50 Hertz**, to **few thousands Hertz**.

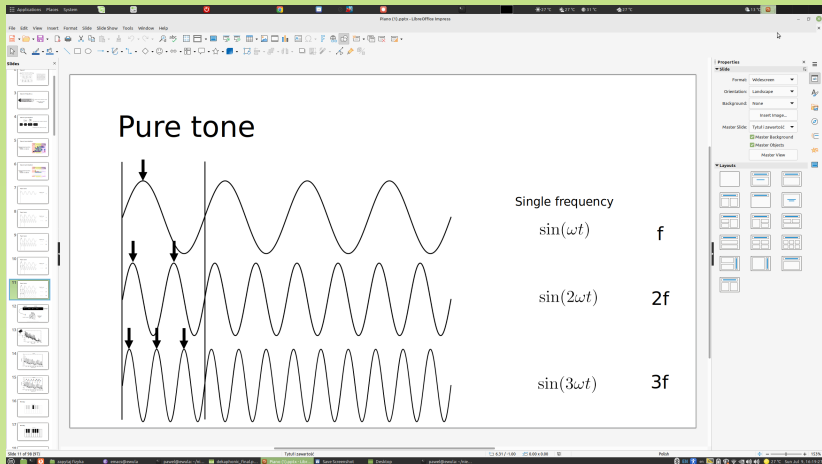
High frequency \iff high pitch

- Our brain perceives higher frequencies detected by our ears as higher pitch
- [▶ Link](#)

Pure tones

- All sounds are made, in a specific way, by building blocks, namely by the **pure tones**.
- We model a pure tone with sound frequency f , by a function *sinus*, which changes with time t as $\sin(2\pi f t)$.

Sines of pure tones with various frequencies

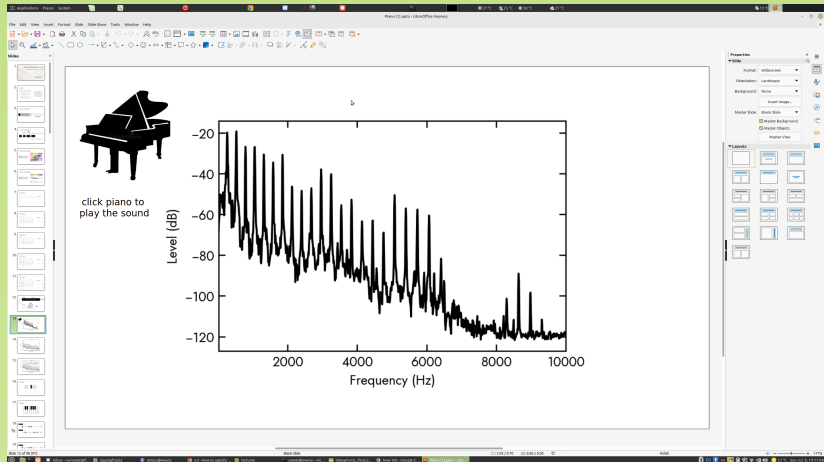


- Relation between ω and f is $\omega = 2\pi f$.

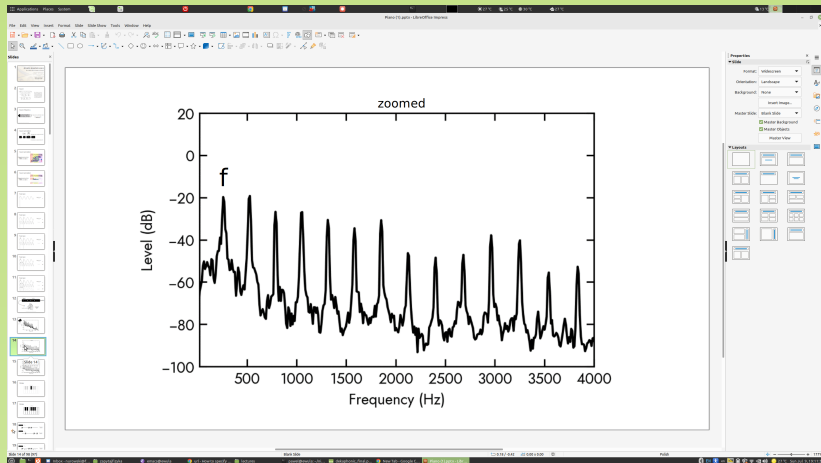
- When we **force a musical instrument to play a pure tone with frequency f** , an (actual infinite) **number of other pure tones, with frequencies $2f$, $3f$, $4f$, etc**, called **overtones**, is created.

▶ [Link](#)

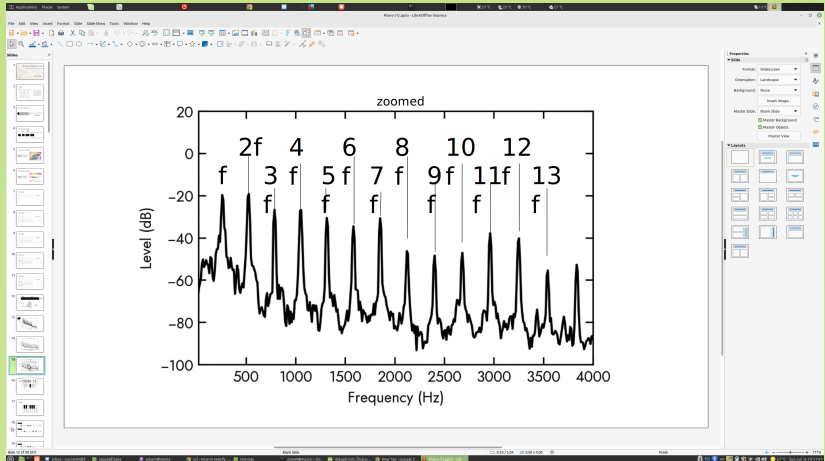
Tones and overtones for a piano



Tones and overtones for a piano

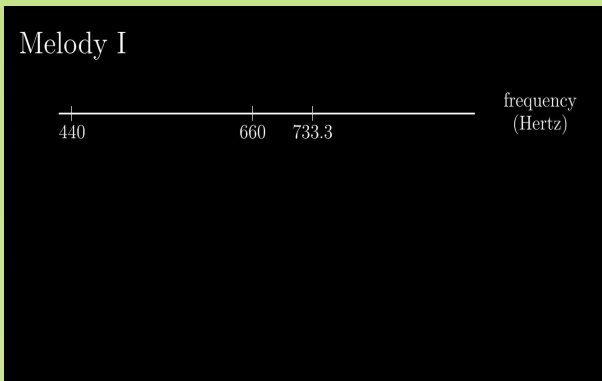


Tones and overtones for a piano



What is a melody?

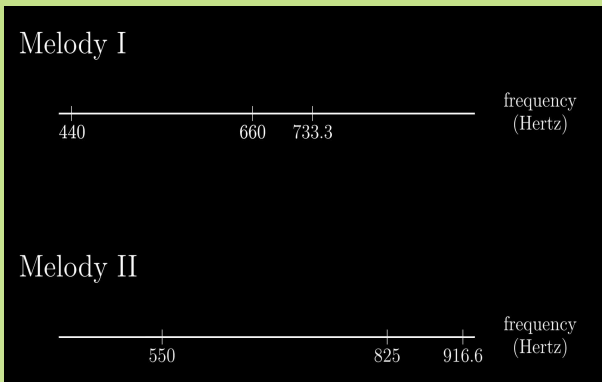
- For a melody, we need more than one tone.



Two, or one melody?

- [▶ Link](#)
- What makes these into melodies? Answer: the change in pitch.
- Are these melodies the same? Strictly speaking NO, since none of the notes of the first melody is the same as a note of the second melody.
- But we strongly feel that these melodies are the same.

What is a melody?



- Different frequencies, but melodies are the same! Why?

What is a melody?

Melody I

$$\frac{660}{440} = \frac{3}{2}$$

440

660

733.3

frequency
(Hertz)

Melody II

550

825

916.6

frequency
(Hertz)

What is a melody?

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frequency
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Melody II



frequency
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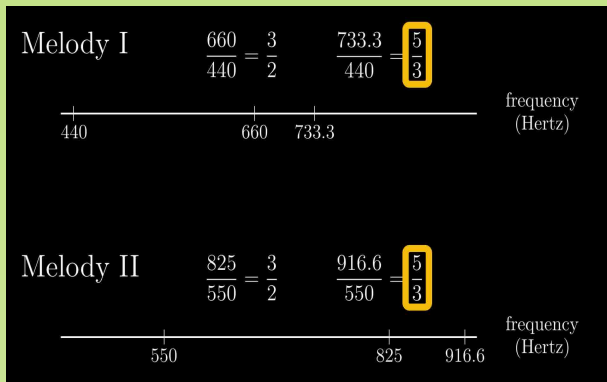


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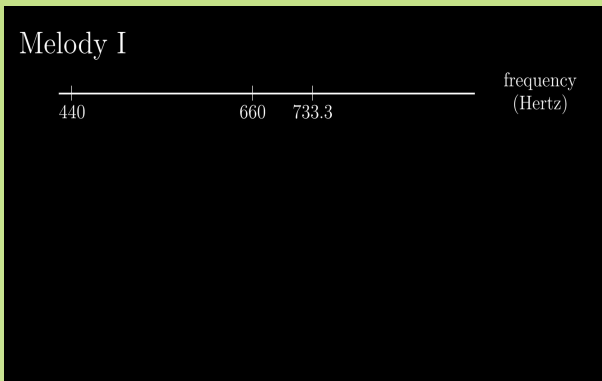


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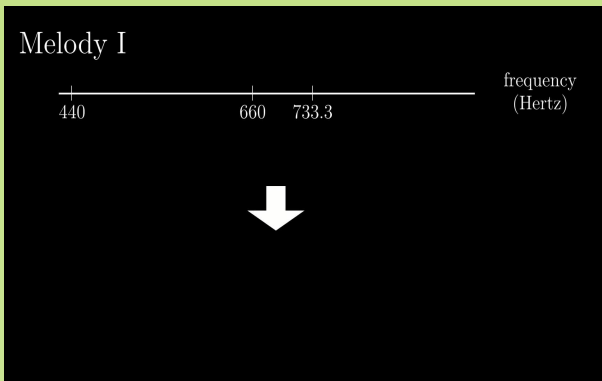


Same frequency \Leftrightarrow same melody!

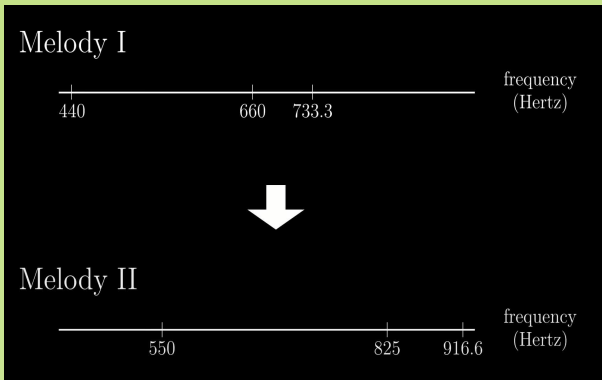
- These are the **ratios of neighbouring frequencies** which **are the same!**
- Conclusion:
the same melodies \Leftrightarrow same ratios between the keys



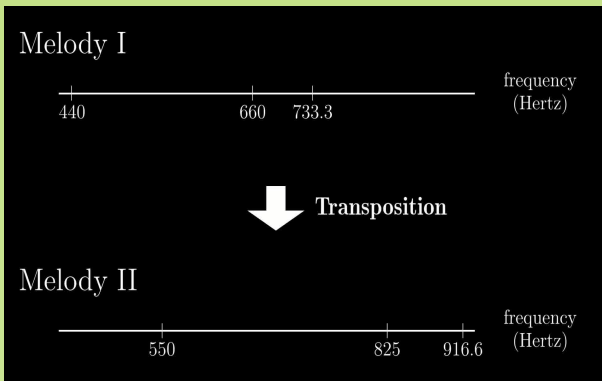
The act of changing of all the frequencies of a given melody by the same factor - so that the ratios between the frequencies stay the same - is called **transposition**.



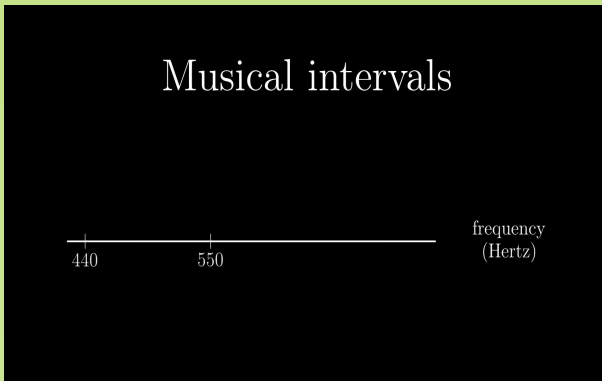
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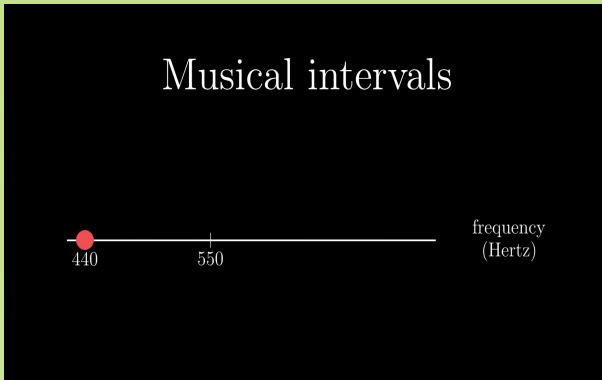
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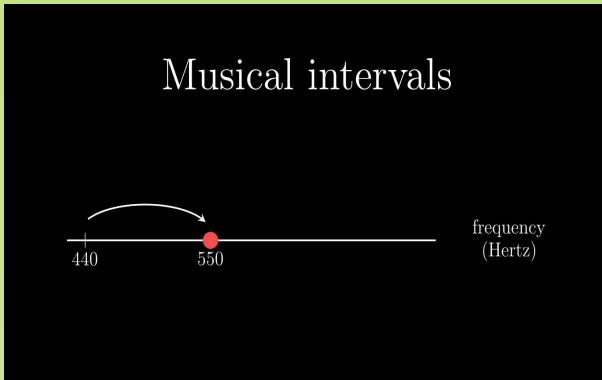
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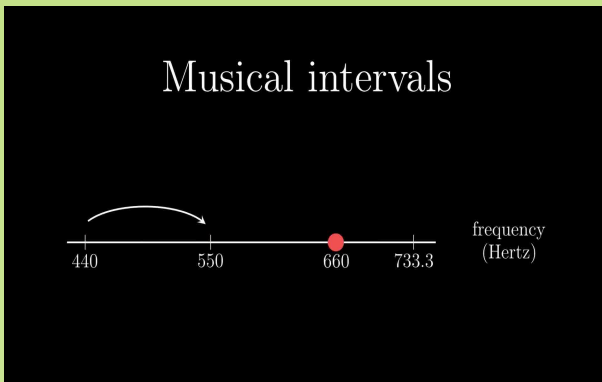
- Between any two tones there is a **musical interval**



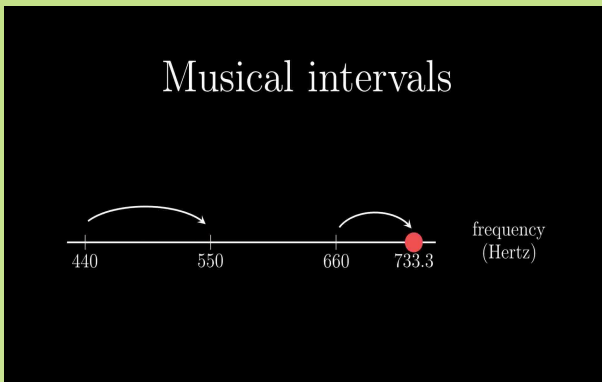
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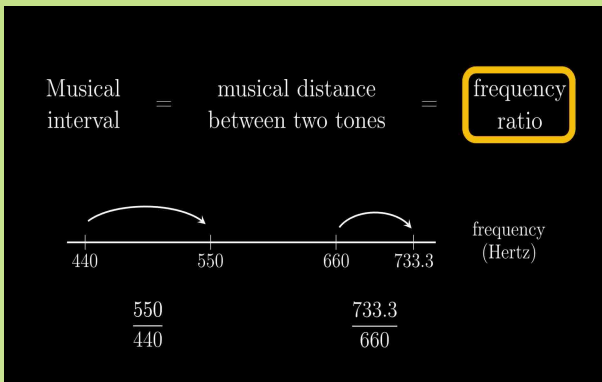


- musical interval = musical distance between two tones



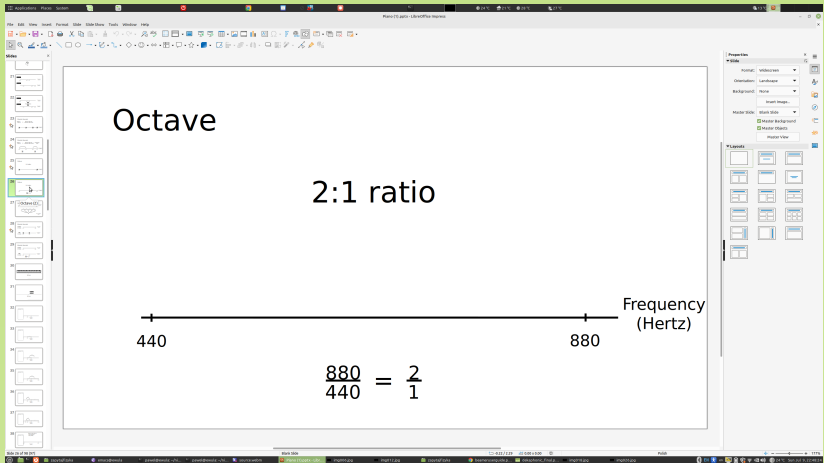
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Musical intervals



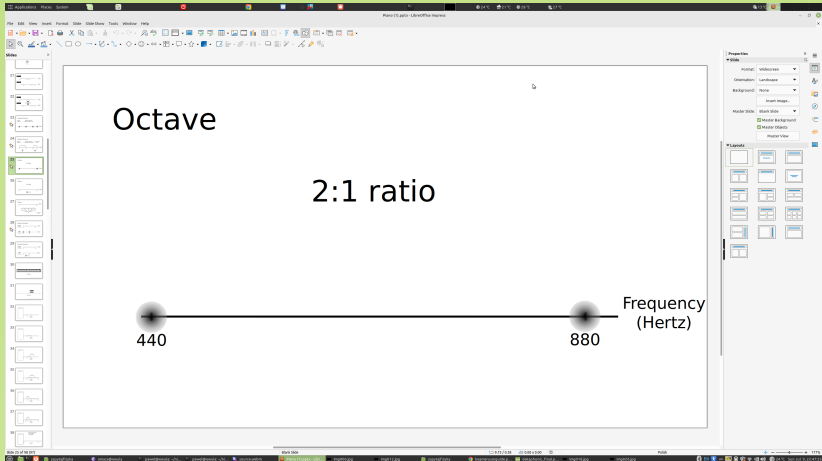
- We now introduce two important intervals.

The octave



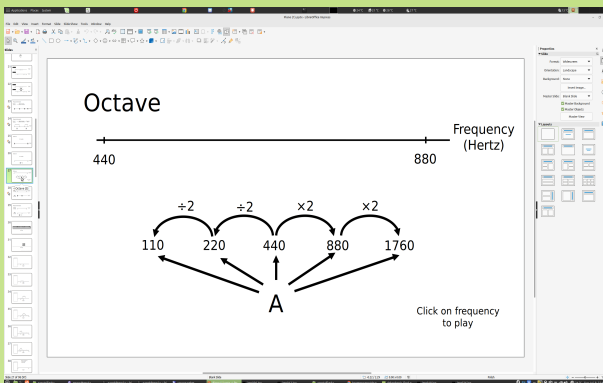
- **The octave = musical interval with ratio 2:1**

The octave

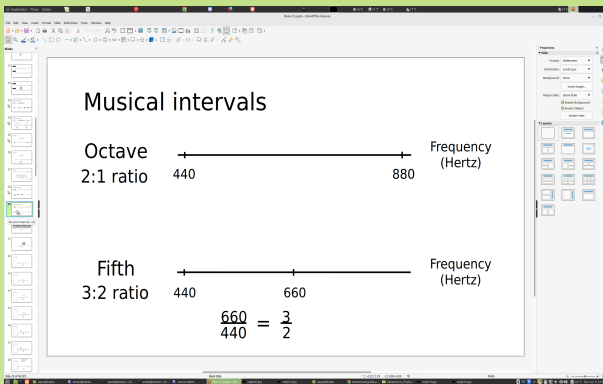


- The **octave** is very pleasant to the ear.
- Two tones, an **octave apart**, played together sound **very harmonious**

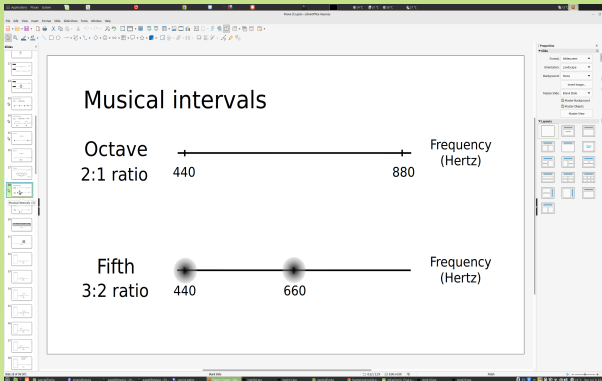
The octave equivalence



- They are **highly similar**; they are so similar that musicians denote them by **the same letter**, and musicologists say that they belong to the same **pitch class**.
- They are considered to be musically **equivalent**.
- We say about the **octave equivalence**.



- Another important **musical interval** is given by the **ratio 3:2**.
- It is called **the fifth**.



- Another important **musical interval** is given by the **ratio 3:2**.
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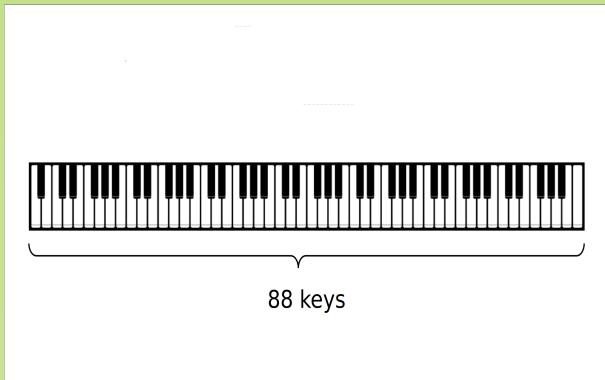
- Due to the **octave equivalence**, the **higher harmonics**

$$2f, 3f, 4f, 5f, 6f, 7f, 8f, 9f, 10f, 11f, 12f, 13f, \dots,$$

can be **placed between f and $2f$** .

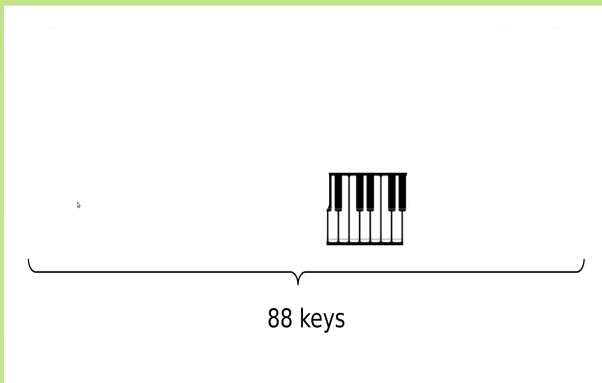
- Since for melodies only the **ratios between the tones** are important, so $f, 2f, 4f, \dots, 2^k f$ are **octave equivalent**, as well as $\frac{1}{2}f, \frac{1}{4}f, \dots, \frac{1}{2^k}f$, for every integer k .
- For example the next musical interval between the harmonics after f and $2f$ is f and $3f$. But this, due to the octave equivalence is the same, as an interval between f and $\frac{3}{2}f$, which explains **pleasance of the fifth**. It is the **next** harmonious musical interval to consider, **after the octave**.

The piano keyboard



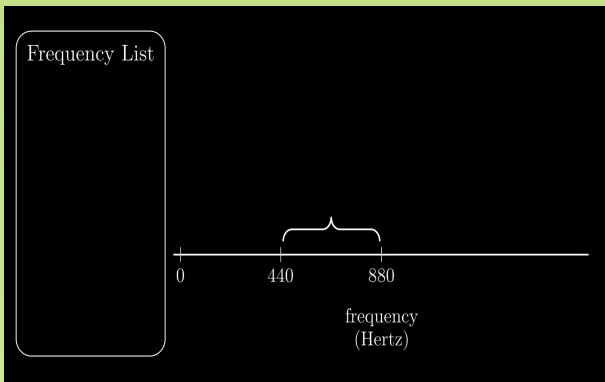
- We are in a position to choose pleasant sounds for the piano keys.
- The piano keyboard has $2 + 12 \times 7 + 2 = 88$ keys.

The piano keyboard

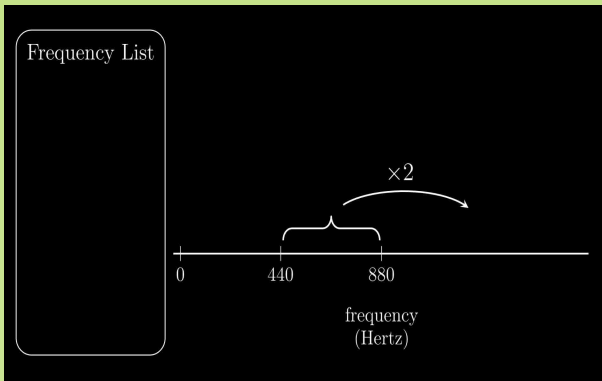


- Look at the above above patern of **twelve** keys. In the keyboard, the pattern is repeated **seven** times, an octave apart from each other. This gives $12 \times 7 = 84$ keys. To these 84 keys, two additional keys are added on each of the lateral ends of the keyboard.

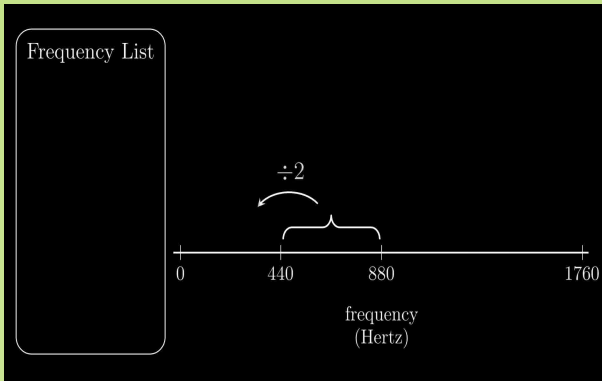
- Which tones are chosen for the **twelve** keys in an octave?



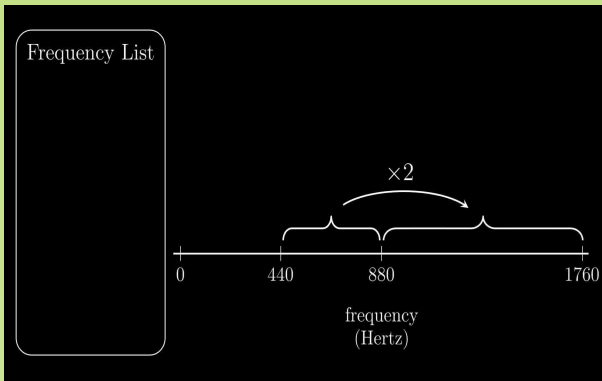
- We start with an octave, say 440Hz – 880Hz.
- We can restrict ourselves to choose frequencies in one octave only, because frequencies for the keys in other octaves will be obtained either



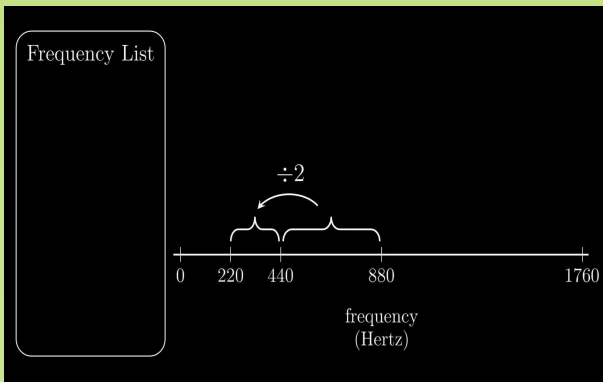
- by **multiplying** all the frequencies by 2, or



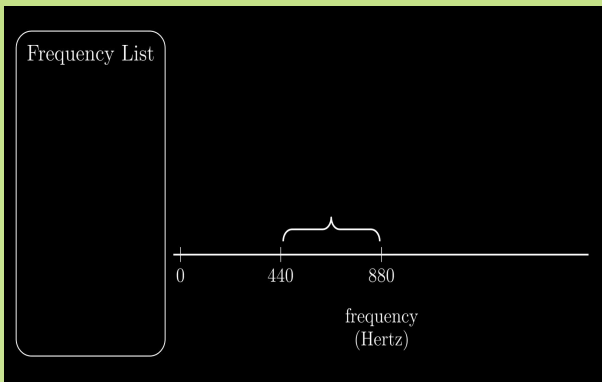
- by **dividing** frequencies by 2.



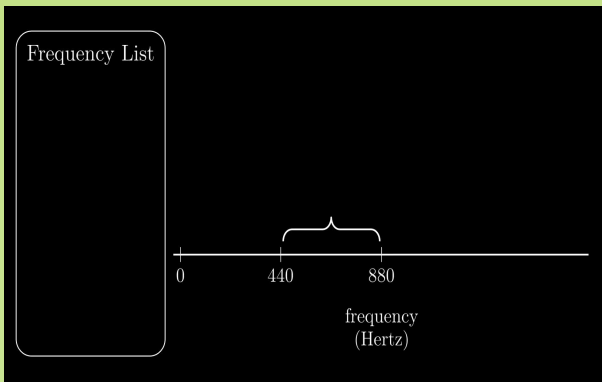
- For example, the **next octave** will have the frequency range: 880Hz–1760Hz,



- and the **previous octave** will have the frequency range: 220Hz–440Hz.

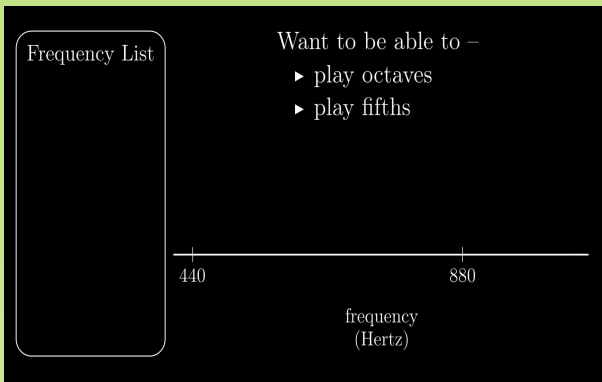


- We will now make frequency choices for the keys in our chosen octave 440Hz-880Hz.

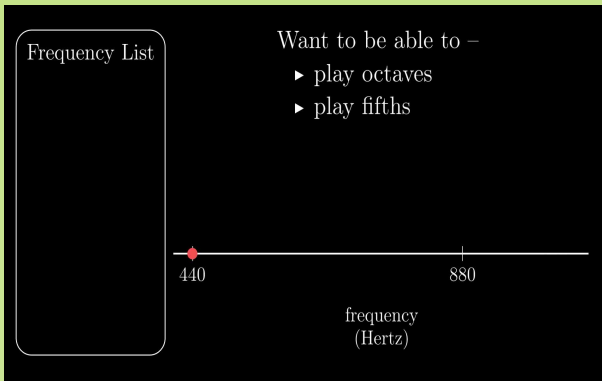


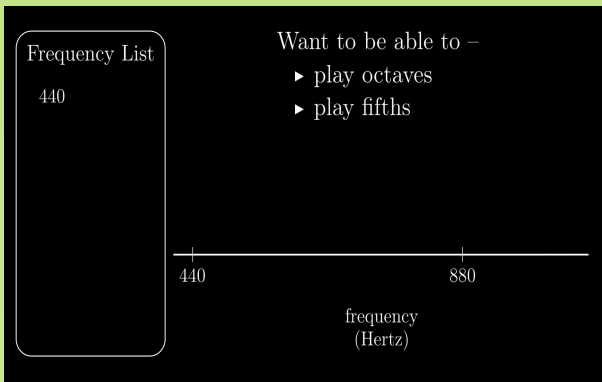
- Of course we **want to play octaves**, so we have to have two frequencies octave apart in our frequency list.
- We also **want to play fifths**, so the corresponding frequency should also be in our list.

Here is as it goes:

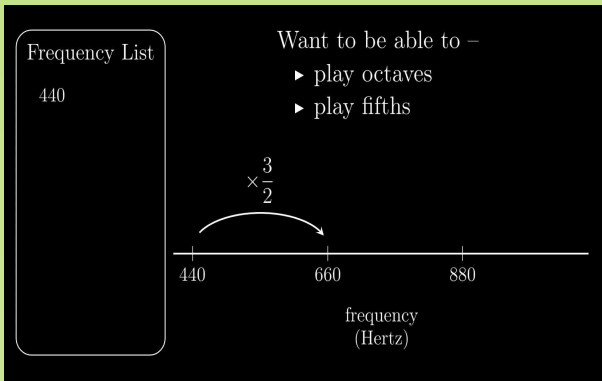


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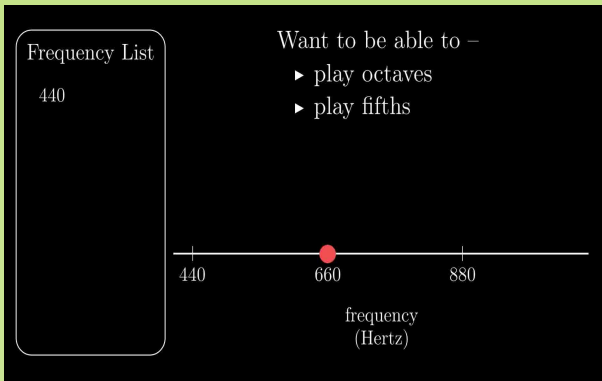


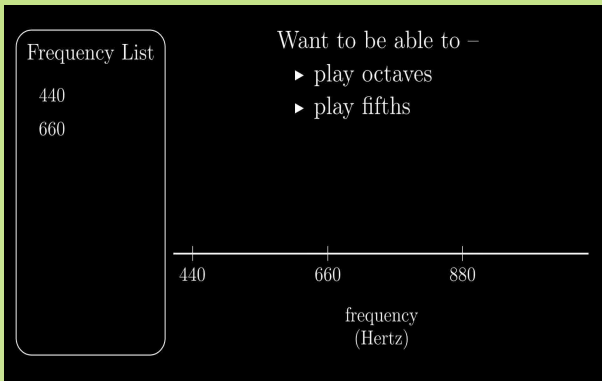


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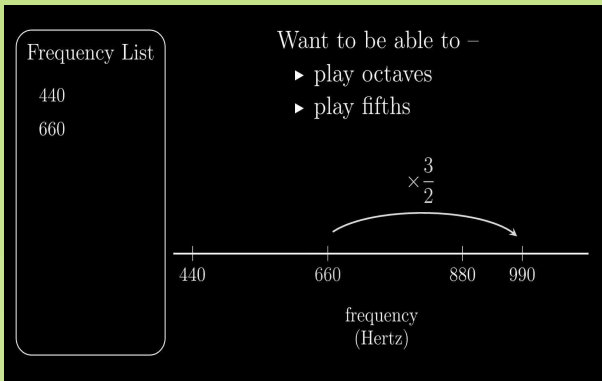


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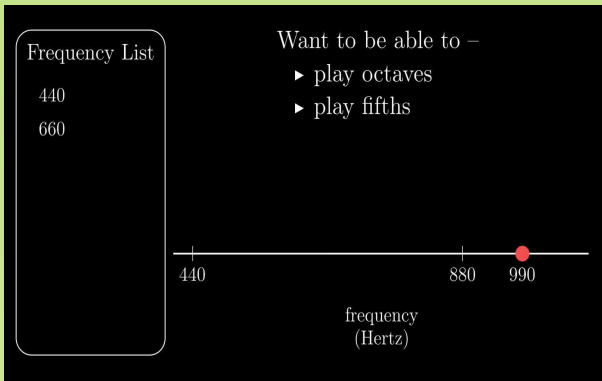




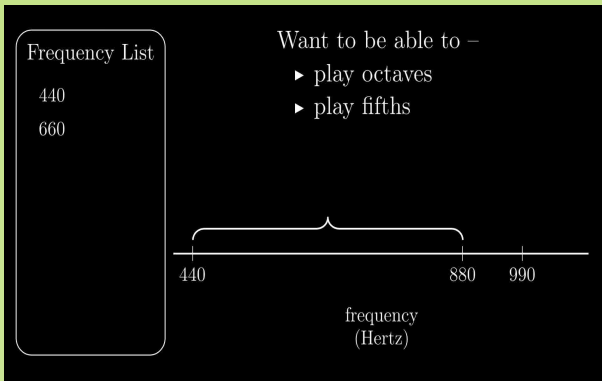
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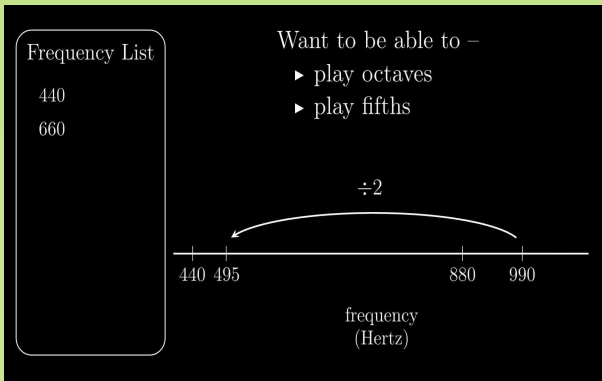
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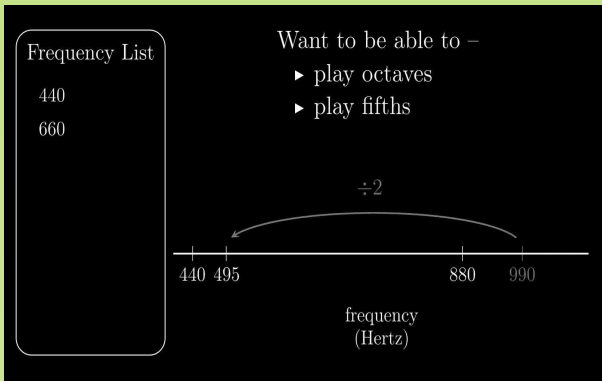
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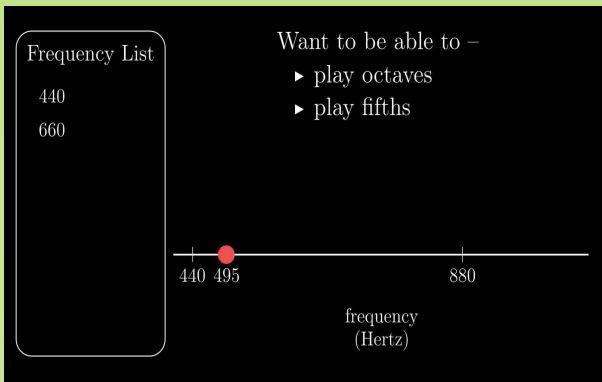
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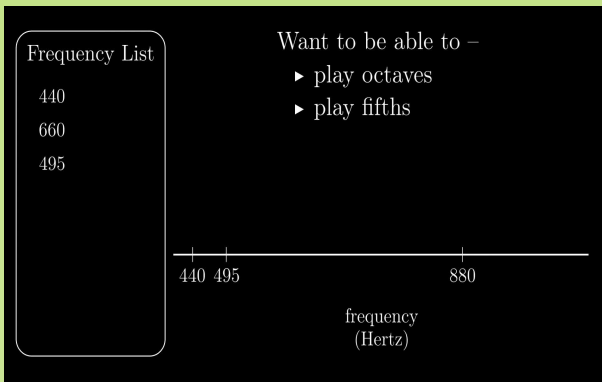
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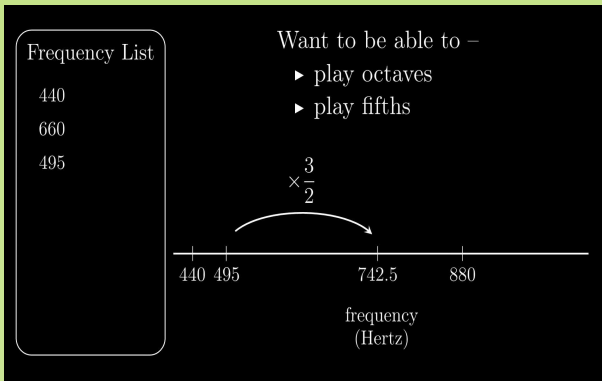
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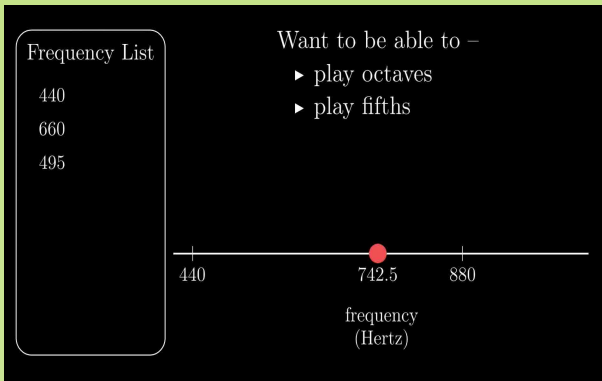
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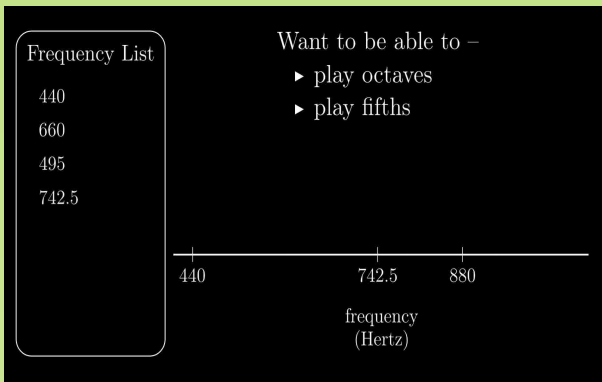
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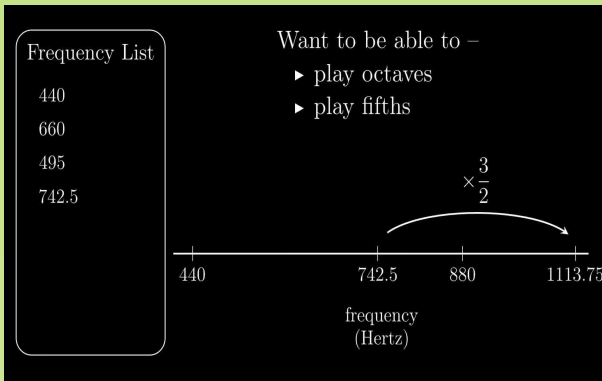
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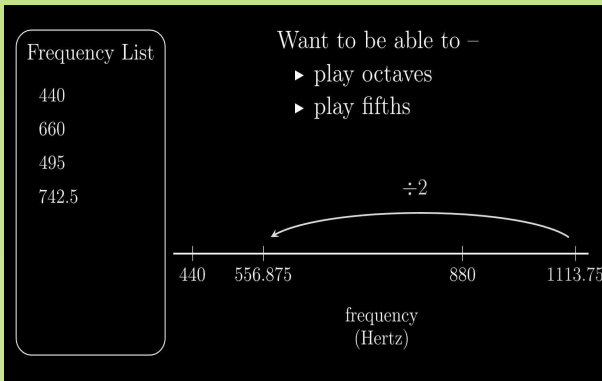
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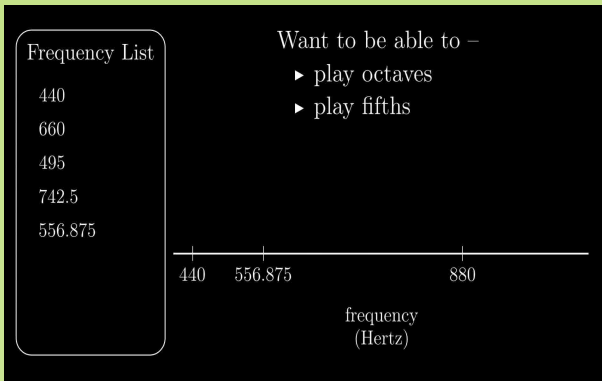
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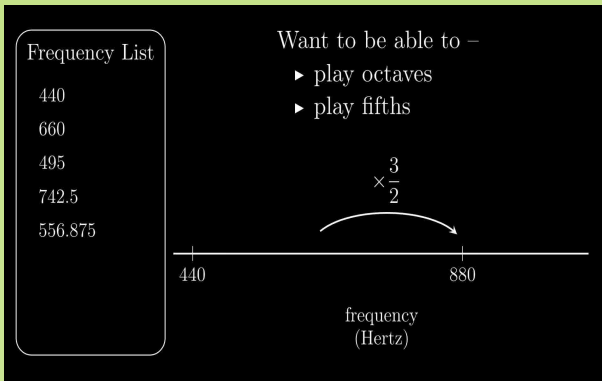


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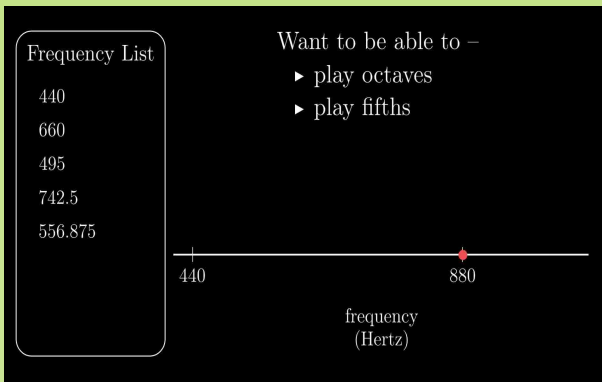
- When does this stop?

When does it stop?

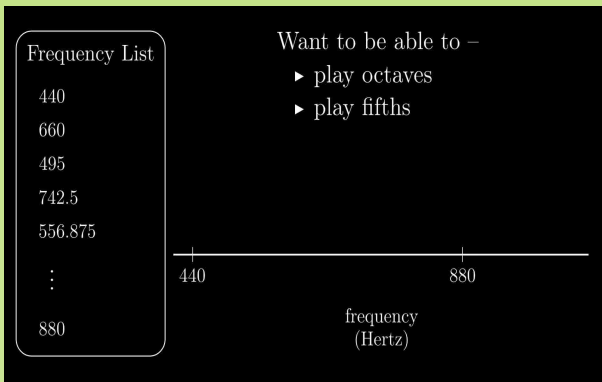


- Well... the best would be if we eventually arrived at 880Hz.

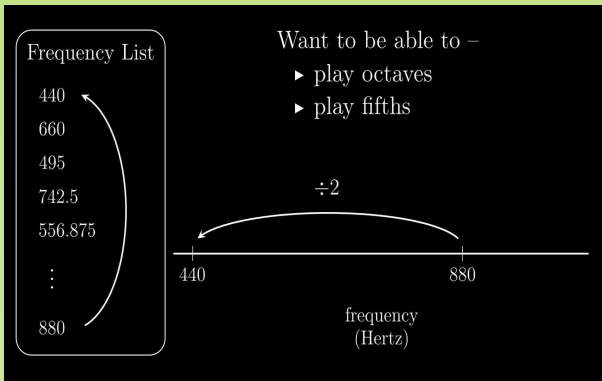
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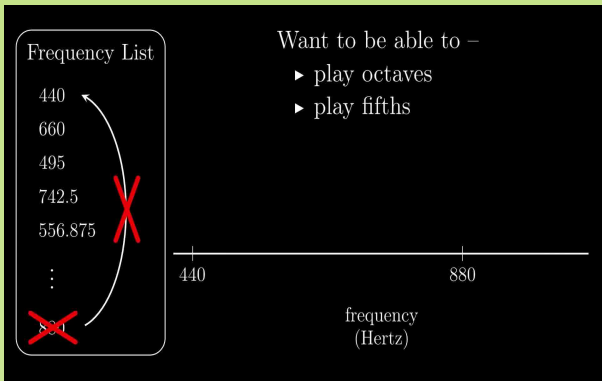


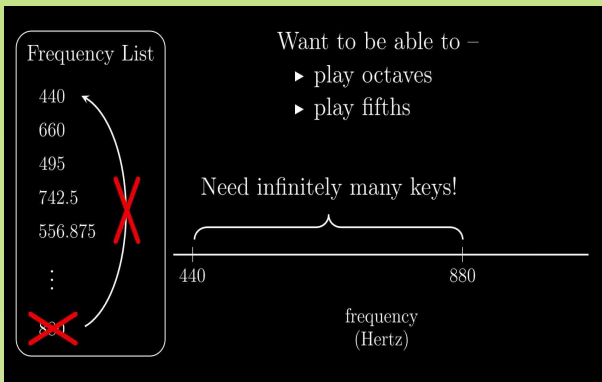
When does it stop?



When does it stop?







If get back to 440 –

n = no. of times went up a fifth

k = no. of times went down an octave

$$440 \cdot \left(\frac{3}{2}\right)^n \cdot \left(\frac{1}{2}\right)^k = 440$$

If get back to 440 –

n = no. of times went up a fifth

k = no. of times went down an octave

$$\cancel{440} \cdot \left(\frac{3}{2}\right)^n \cdot \left(\frac{1}{2}\right)^k = \cancel{440}$$

$$3^n = 2^{n+k}$$

If get back to 440 –

n = no. of times went up a fifth

k = no. of times went down an octave

$$\cancel{440} \cdot \left(\frac{3}{2}\right)^n \cdot \left(\frac{1}{2}\right)^k = \cancel{440}$$

$$\text{odd number} = 3^n = 2^{n+k} = \text{even number}$$

contradiction

Want to be able to –

- ▶ play octaves
- ▶ play fifths



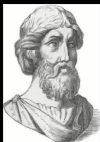
Need a piano with
infinitely many keys

Want to be able to –

- ▶ play octaves
- ▶ play fifths



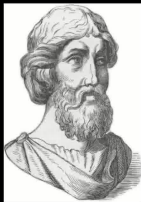
Need a piano with
infinitely many keys



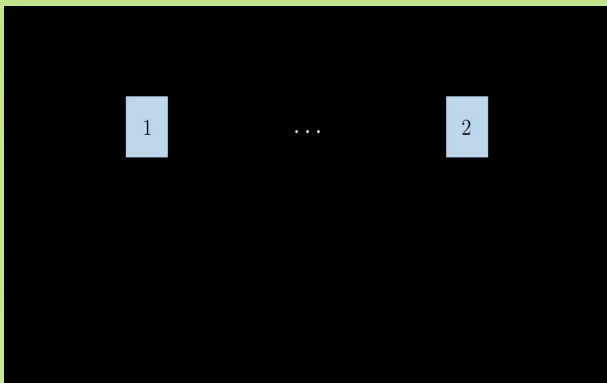
A choice of note frequencies for an octave –
“tuning”,
“intonation”,
“temperament”

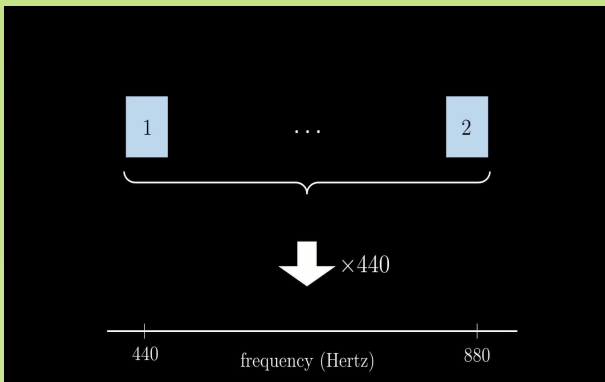
- There are numerous choices of tuning systems!

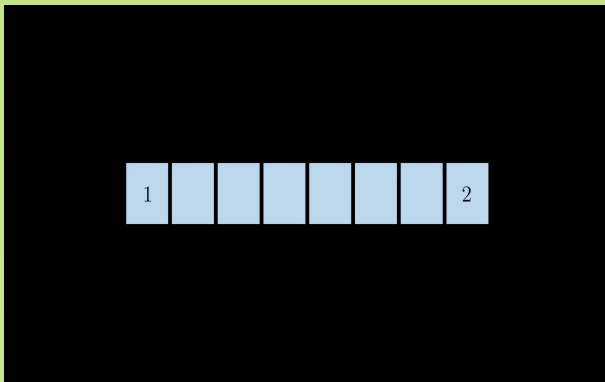
Pythagorean Tuning

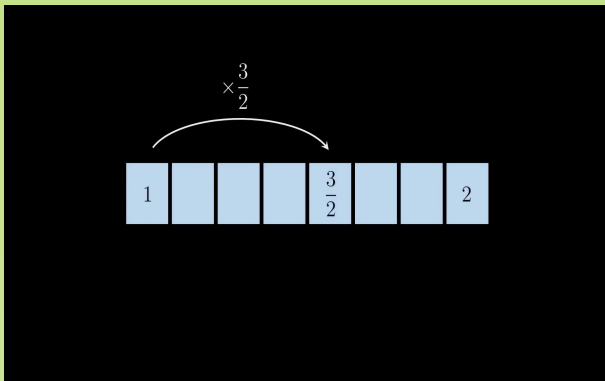


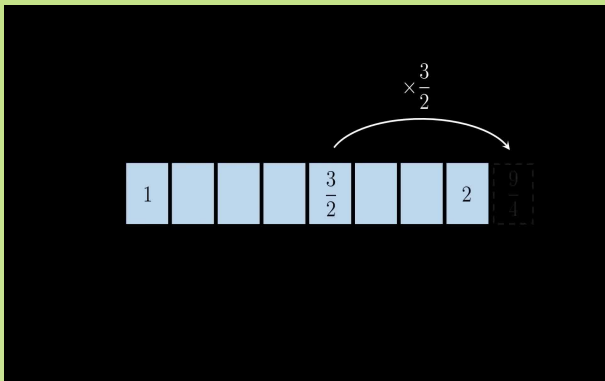
Based on octaves and fifths,
but just a few



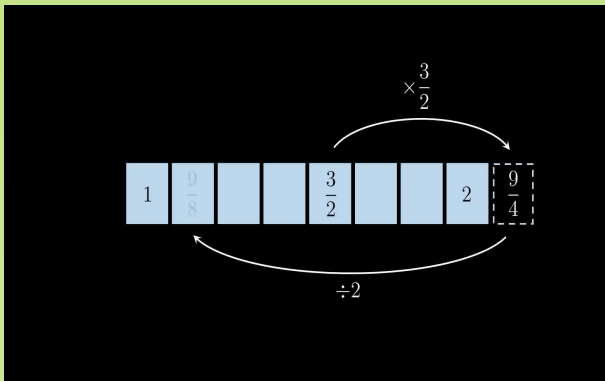


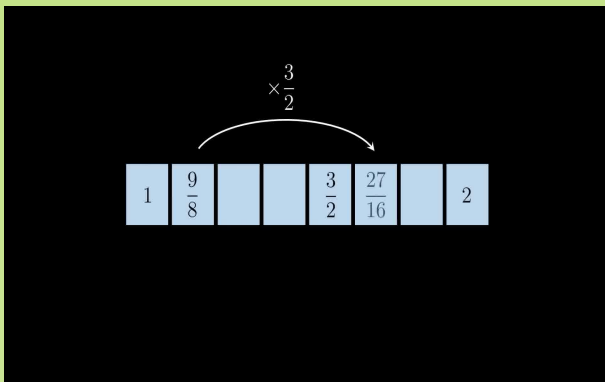


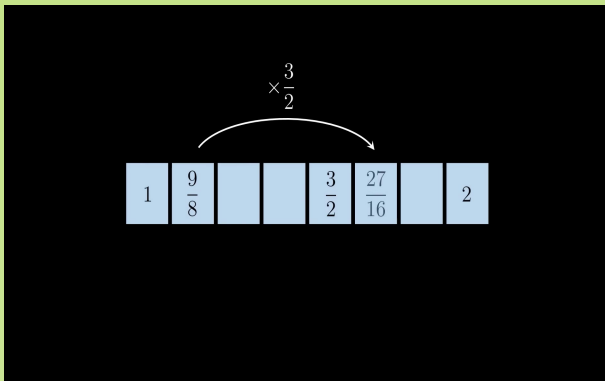


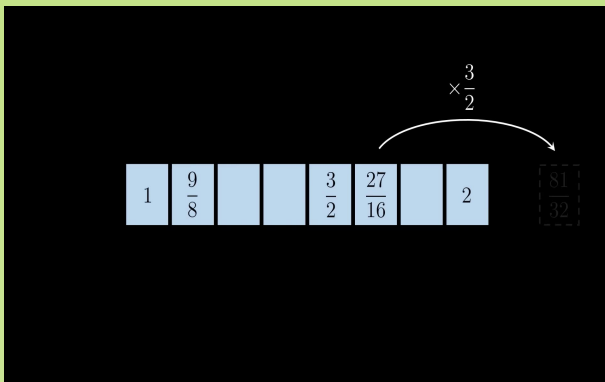


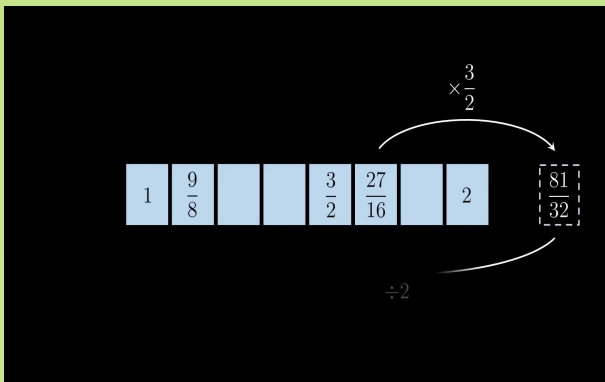
Pythagorean tuning

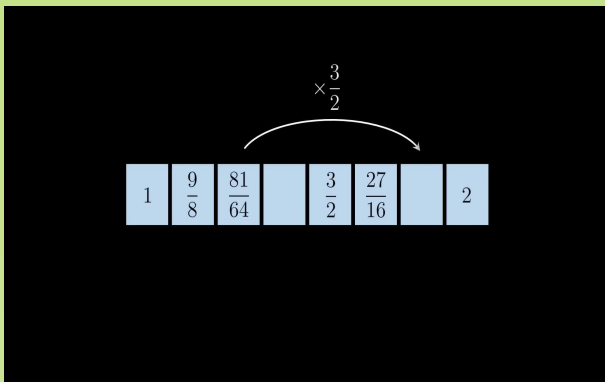


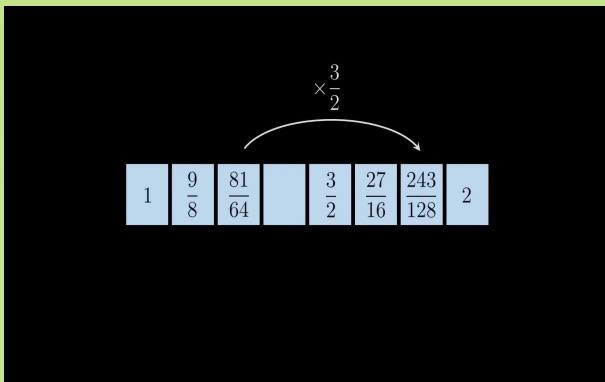




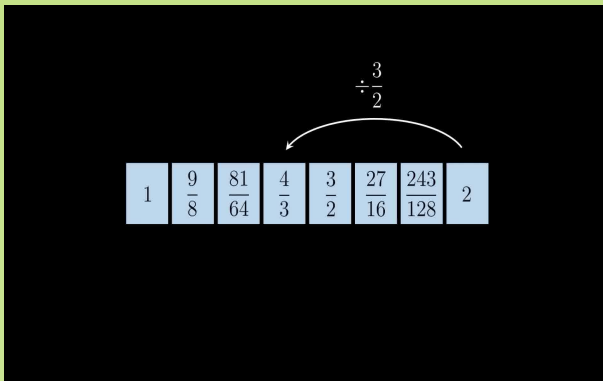








1	$\frac{9}{8}$	$\frac{81}{64}$		$\frac{3}{2}$	$\frac{27}{16}$	$\frac{243}{128}$	2
---	---------------	-----------------	--	---------------	-----------------	-------------------	---



- Uff!...we assigned frequencies for the seven keys.
- How do they sound?

▶ Link

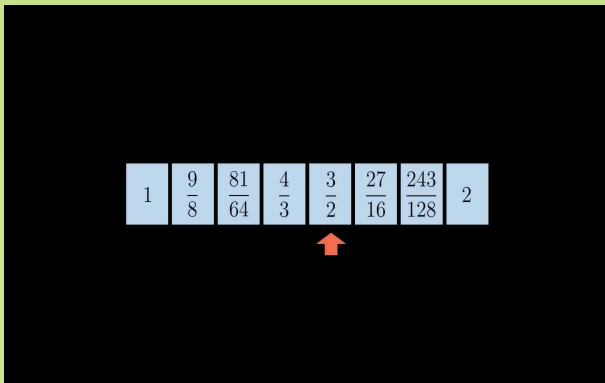
- Sounds familiar, ha?
- It is enough to play a simple melody.

Pythagorean tuning: simple melody

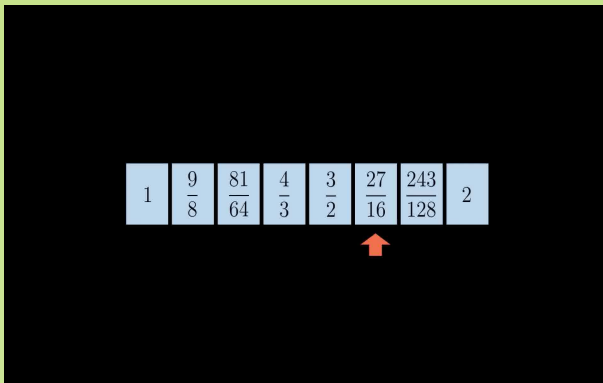
▶ [Link](#)

Pythagorean tuning: can we transpose?

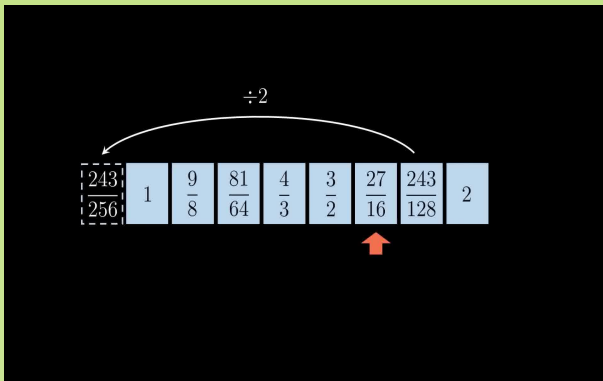
- There is however, a problem with a transposition: We started our melody from the fifth, i.e. $\frac{3}{2}$ key. If we started from the next key, $\frac{27}{16}$, we would be missing one note, to play the same melody :(.



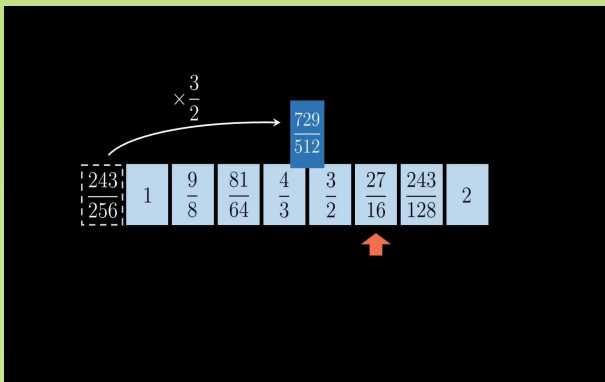
- There is however, a problem with a transposition: We started our melody from the fifth, i.e. $\frac{3}{2}$ key. If we started from the next key, $\frac{27}{16}$, **we would be missing one note**, to play the same melody :(.



Pythagorean tuning: problems with transposition

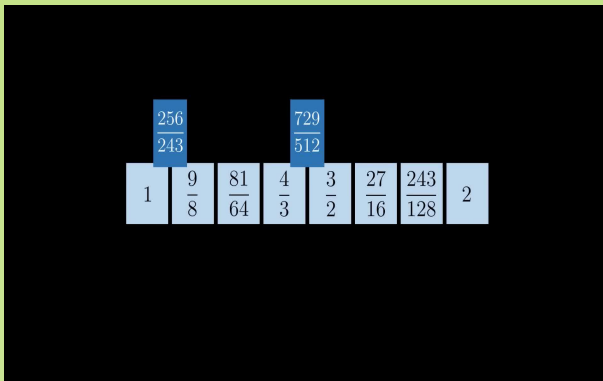


Pythagorean tuning: problems with transposition

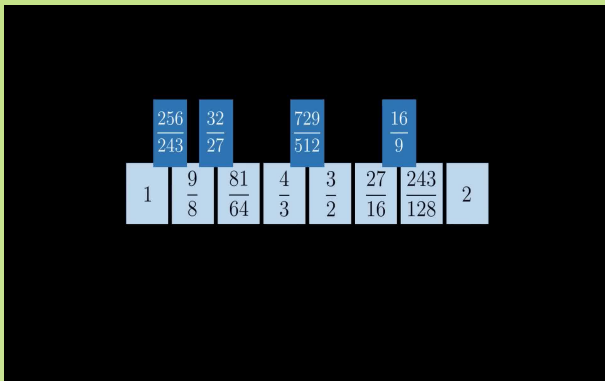


- We need to add only one key more, we again can play the same melody, now starting at $\frac{27}{16}$.
- [▶ Link](#)

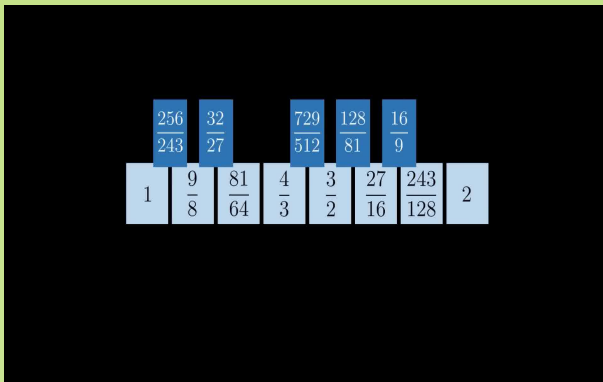
Pythagorean tuning: creating the black keys



Pythagorean tuning: creating the black keys

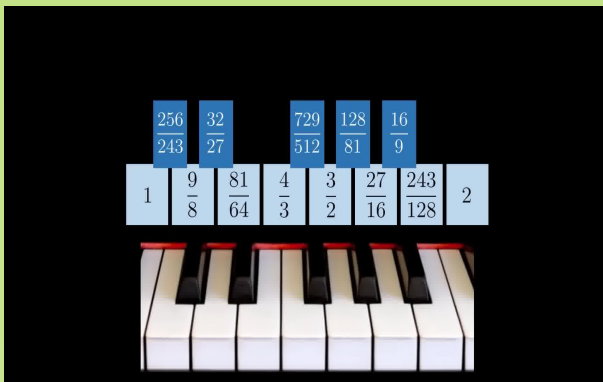


Pythagorean tuning: creating the black keys

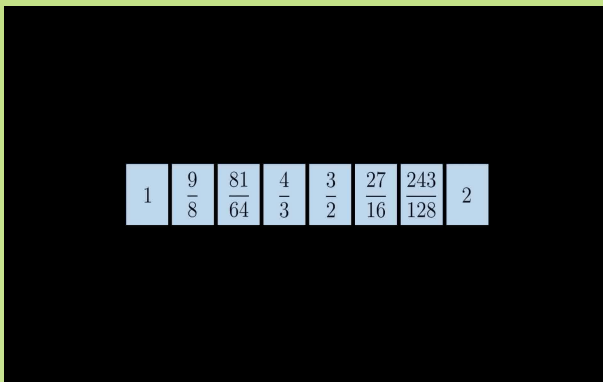


- Uff...We have 12 keys! 7 whites, and 5 blacks!

Pythagorean tuning: full keyboard



- What we have now is pretty much as is the fundamental pattern of the twelve keys of the piano keyboard.
- The Pythagorean system, has more trouble issues than just having troubles with transposition. The **just intonation** -yet another tuning system - is introduced to cure one of them.



1	$\frac{9}{8}$	$\frac{81}{64}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{27}{16}$	$\frac{243}{128}$	2
---	---------------	-----------------	---------------	---------------	-----------------	-------------------	---

- One problem of the Pythagorean tuning is caused by the **beats**.
- Some important **chords** - the multiple harmonious tones played at once - sound better in **just intonation**.

1	$\frac{9}{8}$	$\frac{81}{64}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{27}{16}$	$\frac{243}{128}$	2
---	---------------	-----------------	---------------	---------------	-----------------	-------------------	---

- In particular, instantaneous play of the **1st, 3rd and 5th note**, in the white keys, is an important chord, called **the major chord**.
- [▶ Link](#)
- And the problem is with the **5th overtone** of the first note, and the **4th overtone** of the 3rd note.

1	$\frac{9}{8}$	$\frac{81}{64}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{27}{16}$	$\frac{243}{128}$	2
---	---------------	-----------------	---------------	---------------	-----------------	-------------------	---

- It is as simple as:

$$4 \times \frac{81}{64} \simeq 5.0625 \simeq 5 \times 1.$$

- Because a superposition of two sinusoidal waves with so close frequencies produces this:

- [▶ Link](#)

Just	1	$\frac{9}{8}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{15}{8}$	2
Pyth.	1	$\frac{9}{8}$	$\frac{81}{64}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{27}{16}$	$\frac{243}{128}$	2

$$\frac{81/64}{5/4} = 1.0125$$

Just	1	$\frac{9}{8}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{15}{8}$	2
Pyth.	1	$\frac{9}{8}$	$\frac{81}{64}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{27}{16}$	$\frac{243}{128}$	2

$$\frac{243/128}{15/8} = 1.0125$$

Just	1	$\frac{9}{8}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{15}{8}$	2
Pyth.	1	$\frac{9}{8}$	$\frac{81}{64}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{27}{16}$	$\frac{243}{128}$	2

$$\frac{27/16}{5/3} = 1.0125$$

Just	1	$\frac{9}{8}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{15}{8}$	2
Pyth.	1	$\frac{9}{8}$	$\frac{81}{64}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{27}{16}$	$\frac{243}{128}$	2

- Long time ago people observed that

$$\left(\frac{3}{2}\right)^{12} \simeq 2^7.$$

- This is the same as saying that

$$\left(\frac{3}{2}\right)^{\frac{12}{7}} \simeq 2.$$

- Well...

$$\left(\frac{3}{2}\right)^{\frac{12}{7}} - 2 \simeq 0.00387547.$$

Circle of fifths

The slide is titled "Circle of fifths" and contains the following content:

$\times \frac{3}{2} \quad \times \frac{3}{2} \quad \times \frac{3}{2} \quad \dots \quad \times \frac{3}{2}$

Below the sequence, a horizontal line with arrows indicates the progression of the sequence.

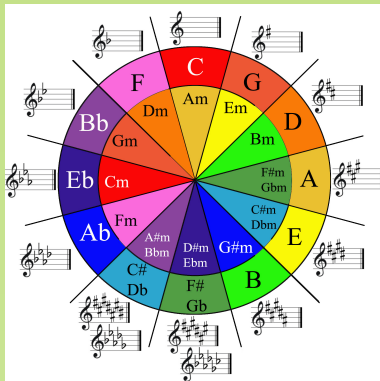
$\frac{3}{2} \times \frac{3}{2} \times \dots \times \frac{3}{2} = \left(\frac{3}{2}\right)^{12} \approx 129.746 \approx 128 = 2^7$

12 fifths **7** octaves

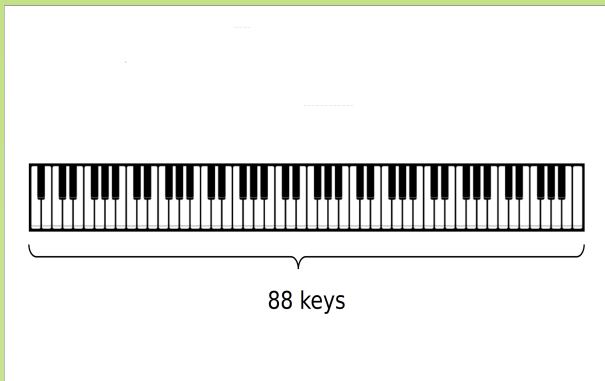
The slide is displayed in a LibreOffice Impress window. The Properties panel on the right shows slide settings: Format: Widescreen, Orientation: Landscape, Background: None, Master Slide: Blank Slide, and Layouts: Blank Slide.

- This means that starting in the pitch class **C** and heating the successive key notes by the musical intervals of the **fifth**, the thirteenth heated fifth will sound as **C**.
- Well..., with a good approximation.

Circle of fifths



► Link



- This, in particular explains $88 = 2 + 12 \times 7 + 2$ keys of the piano!
- **twelve** keys in each octaves, and **seven** octaves, to go around in terms of the fifths to the same pitch class *C*.

- For the purpose of this lecture, I will call the **modulus of the difference** $|(\frac{3}{2})^{\frac{12}{7}} - 2| = \mu_{(3,12,7)}$, a **comma**, of the tuning system.
- In the tuning systems we considered here, **the comma** $\mu_{(3,12,7)} = 0.00387547$ is related to the **three natural numbers** f, t, k which are:
 - $f = 3$, the third harmonic, which we used in the form of the ratio $\frac{3}{2}$, to generate our piano keys steps;
 - $t = 12$, the number of keys in one octave of our tuning system;
 - $k = 7$, the number of octaves needed to go from C back to C , by jumping through the keys by the intervals of $\frac{3}{2}$.

- **Question:** Can we have a Pythagorean tuning system with t keys in every octave, which has k octaves and which is generated from the first octave $1 \longleftrightarrow 2$ by a harmonic² f , such that
 - its comma $\mu_{(f,t,k)} < 0.01$,
 - the number of tones $t \times k < 100$,
 - its generating harmonic $f < 64$?
- I call this system Pythagorean, although it is based on f rather than $f = 3$. But this is a straightforward generalization.

²or a subharmonic $1/f$

Theorem

The only values of (f, t, k) that answer the question in positive are in the following table:

f	t	k	μ
$\frac{3}{2}$	12	7	0.00387547
$\frac{11}{8}$	13	6	0.00631819
$\frac{13}{8}$	10	7	0.000871016
$\frac{17}{16}$	23	2	0.00808825
$\frac{27}{16}$	4	3	0.00905446
$\frac{29}{16}$	7	6	0.00135602

Designing a tuning system

f	t	k	μ
$\frac{33}{32}$	45	2	0.00156911
$\frac{37}{32}$	19	4	0.0070528
$\frac{39}{32}$	7	2	0.00151358
$\frac{41}{32}$	14	5	0.00158879
$\frac{43}{32}$	7	3	0.00744747
$\frac{47}{32}$	9	5	0.00241079
$\frac{53}{32}$	11	8	0.00123505
$\frac{55}{32}$	9	7	0.00639457
$\frac{57}{32}$	6	5	0.000737349
$\frac{59}{32}$	9	8	0.00971721

- We see that among all Pythagorean systems, based on $f < 56$, the system with $f = 13$, $t = 10$, $k = 7$ is the system with the **smallest comma**.
- It is precisely the system which **Leszek Mozdér** wanted us to design for him. His motivation for using $t = 10$, $k = 7$ is kind of 'mystic'; totally unclear for us.
- The Pythagorean **decaphonic piano** has **six white** keys and **four black** keys.
- Its octave with **six white** keys is generated by $f = \frac{13}{8}$, which plays the role of the 'fifth', when compared to the $t = 12$, $k = 7$ Pythagorean system.
- Since we have **ten** keys, and $k = 7$, to cover a passage from C to C via the 'fifths' of $\frac{13}{8}$, the piano needs 70 keys, only.
- It should be stressed, that in our analysis we did not insist on having $t = 10$ keys! The **decimal/decaphonic** system, with **seven** octaves, was distinguished by pure mathematics.

Ten scale Pythagorean tuning

The image shows a screenshot of the LibreOffice Impress application window. The title bar reads "Rass_v2 (1).pptx - LibreOffice Impress". The menu bar includes "File", "Edit", "View", "Insert", "Format", "Slide", "Slide Show", "Tools", "Window", and "Help". The toolbar contains various icons for editing and presentation control. On the left, a "Slides" pane shows a list of slide thumbnails, with slide 97 selected. The main slide area displays the text "Let's create all steps of 10-step scale". On the right, the "Properties" pane is open, showing settings for the "Slide" (Format: Widescreen, Orientation: Landscape, Background: None) and "Layouts" (a grid of layout thumbnails). The status bar at the bottom indicates "Slide 97 of 129 (1:26)", "Blank Slide", "4:22 / 6:30", "0.00 x 0.00", "Polish", and "Mon Jul 10, 19:43:21".

Ten scale Pythagorean tuning

The screenshot shows a presentation slide titled "Ten scale Pythagorean tuning". The slide content consists of three green boxes arranged horizontally. The leftmost box contains the number "1". The middle box contains the fraction $\frac{13}{8}$. The rightmost box contains the number "2". A curved black arrow originates from the top of the "1" box and points to the top of the $\frac{13}{8}$ box. Above this arrow is the text $\times \frac{13}{8}$, indicating a multiplication operation. The presentation software interface is visible, including a menu bar (File, Edit, View, Insert, Format, Slide, Slide Show, Tools, Window, Help), a toolbar with various icons, a slide thumbnails pane on the left, and a properties panel on the right. The status bar at the bottom shows "Slide 98 of 129 (1/20)", "Blank Slide", and "20 °C".

Ten scale Pythagorean tuning

The screenshot shows a presentation slide titled "Ten scale Pythagorean tuning" in the LibreOffice Impress application. The slide features three dark green squares on a white background. The first square on the left contains the number "1". The second square in the middle contains the fraction $\frac{13}{8}$. The third square on the right contains the number "2". A black arrow points from the $\frac{13}{8}$ square to the "2" square, with the text $\times \frac{13}{8}$ positioned above the arrow. The application interface includes a menu bar (File, Edit, View, Insert, Format, Slide, Slide Show, Tools, Window, Help), a toolbar, a slide navigation pane on the left, a Properties panel on the right, and a status bar at the bottom.

Ten scale Pythagorean tuning

The slide illustrates the derivation of the number $\frac{169}{128}$ through a series of operations:

- Start with the number 1.
- Multiply by $\frac{13}{8}$ (indicated by an arrow labeled $\times \frac{13}{8}$).
- Divide by 2 (indicated by an arrow labeled $\div 2$).

The final result is $\frac{169}{128}$.

The slide is displayed in the LibreOffice Impress application window. The interface includes a menu bar (File, Edit, View, Insert, Format, Slide, Slide Show, Tools, Window, Help), a toolbar, a slide navigation pane on the left, a Properties panel on the right, and a status bar at the bottom.

Ten scale Pythagorean tuning

The screenshot shows a presentation slide titled "Blank Slide" in LibreOffice Impress. The slide contains four dark green squares arranged horizontally, each containing a number or fraction representing a scale step in a Pythagorean tuning sequence:

- 1
- $\frac{169}{128}$
- $\frac{13}{8}$
- 2

The interface includes a menu bar (File, Edit, View, Insert, Format, Slide, Slide Show, Tools, Window, Help), a toolbar, a "Slides" panel on the left showing a sequence of 100 slides, and a "Properties" panel on the right with settings for Slide (Format: Widescreen, Orientation: Landscape, Background: None, Master Slide: Blank Slide, Master Background, Master Objects, Master View) and Layouts. The status bar at the bottom indicates "Slide 101 of 123 (12%)", "Blank Slide", "0.79 / 4.46", "0.00 x 0.00", "Polski", and "Mon, 22.10.2014 19:35:43".

Ten scale Pythagorean tuning

The screenshot shows a presentation slide titled "Ten scale Pythagorean tuning" within a LibreOffice Impress window. The slide features four green boxes containing the numbers 1, $\frac{169}{128}$, $\frac{13}{8}$, and 2, arranged horizontally. A curved arrow above the boxes points from the $\frac{13}{8}$ box to the 2 box, labeled with $\times \frac{13}{8}$. A curved arrow below the boxes points from the 2 box back to the 1 box, labeled with $\div 2$. The presentation interface includes a sidebar on the left with slide thumbnails (91-101), a top menu bar, a toolbar, and a right sidebar with "Properties" and "Layouts" panels. The status bar at the bottom indicates "Slide 102 of 123 (12%)", "Blank Slide", and "20°C".

Ten scale Pythagorean tuning

The slide displays the following elements:

- Slide 93:** A table with two columns: the first column contains the number '1', and the second column contains the fraction $\frac{2197}{2048}$.
- Slide 94:** A table with one column containing the fraction $\frac{169}{128}$.
- Slide 95:** A table with one column containing the fraction $\frac{13}{8}$.
- Slide 96:** A table with one column containing the number '2'.

Arrows indicate the relationships between these notes:

- An arrow from slide 93 to slide 95 is labeled with the multiplication $\times \frac{13}{8}$.
- An arrow from slide 96 to slide 93 is labeled with the division $\div 2$.

The presentation interface includes a menu bar (File, Edit, View, Insert, Format, Slide, Slide Show, Tools, Window, Help), a toolbar, a slide navigation pane on the left, a Properties panel on the right, and a status bar at the bottom.

Ten scale Pythagorean tuning

The screenshot shows a presentation slide titled "Ten scale Pythagorean tuning" in a LibreOffice Impress window. The slide features four green boxes containing musical intervals:

- Box 1: $1 \quad \frac{2197}{2048}$
- Box 2: $\frac{169}{128}$
- Box 3: $\frac{13}{8}$
- Box 4: 2

The presentation interface includes a menu bar (File, Edit, View, Insert, Format, Slide, Slide Show, Tools, Window, Help), a toolbar, a slide navigation pane on the left, and a Properties pane on the right. The status bar at the bottom indicates "Slide 106 of 123 (12%)", "Blank Slide", and "20:19 / 4:46".

Ten scale Pythagorean tuning

Slide content:

$$\times \frac{13}{8}$$

1 $\frac{2197}{2048}$ $\frac{169}{128}$ $\frac{13}{8}$ 2

Properties Panel:

- Slide
- Format: Widescreen
- Orientation: Landscape
- Background: None
- Master Slide: Blank Slide
- Layouts: Multiple layout options shown

System Bar:

Slide 105 of 123 (12%) 0.82 / 4.47 0.00 x 0.00 Polub 20 °C 29 °C 29 °C 24 °C 11 % Mon Jul 10, 19:31:47

Ten scale Pythagorean tuning

Slide 100 of 123 (12%)

Blank Slide

0.82 / 4.47 0.00 x 0.00

Polub

19%

Slide 100 of 123 (12%)

Blank Slide

0.82 / 4.47 0.00 x 0.00

Polub

19%

Ten scale Pythagorean tuning

The screenshot shows a presentation slide titled "Ten scale Pythagorean tuning" in LibreOffice Impress. The slide contains five green boxes with musical notation. The first box is labeled "1" and contains the fraction $\frac{2197}{2048}$. The second box contains the fraction $\frac{169}{128}$. The third box contains the fraction $\frac{13}{8}$. The fourth box contains the fraction $\frac{28561}{16384}$. The fifth box is labeled "2". The interface includes a menu bar, a toolbar, a slide navigation pane on the left, a properties pane on the right, and a status bar at the bottom.

Slide 107 of 123 (12%)

Blank Slide

0.82 / 4.47 0.00 x 0.00

Polish

19%

Mon Jul 10, 19:32:07

Ten scale Pythagorean tuning

The screenshot shows a presentation slide titled "Ten scale Pythagorean tuning" in a LibreOffice Impress window. The slide features a central diagram with five green boxes containing numerical ratios, connected by a curved arrow. Above the diagram is the fraction $\frac{13}{8}$ with a division symbol above it. The boxes contain the following ratios:

- Box 1: $\frac{2197}{2048}$
- Box 2: $\frac{169}{128}$
- Box 3: $\frac{13}{8}$
- Box 4: $\frac{28561}{16384}$
- Box 5: 2

A curved arrow points from the top of the $\frac{13}{8}$ box to the top of the 2 box. The presentation interface includes a menu bar, a toolbar, a slide navigation pane on the left, and a properties pane on the right. The status bar at the bottom shows "Slide 108 of 123 (12%)", "Blank Slide", and "0.82 / 4.47 0.00 x 0.00".

Ten scale Pythagorean tuning

The screenshot shows a presentation slide titled "Blank Slide" in LibreOffice Impress. The slide content includes:

- A central diagram of a circle of fifths with a curved arrow pointing from the 13/8 interval to the 16/13 interval.
- Four green boxes containing musical intervals:
 - Box 1: 1 and $\frac{2197}{2048}$
 - Box 2: $\frac{16}{13}$ and $\frac{169}{128}$
 - Box 3: $\frac{13}{8}$ and $\frac{28561}{16384}$
 - Box 4: 2
- A central equation: $\frac{13}{8} \div \frac{16}{13}$

The interface includes a menu bar (File, Edit, View, Insert, Format, Slide, Slide Show, Tools, Window, Help), a toolbar, a slide navigation pane on the left, and a Properties pane on the right. The status bar at the bottom shows "Slide 109 of 129 (12%)", "Blank Slide", "0.82 / 4.47", "0.00 x 0.00", "Polish", and "Mon Jul 16, 19:32:20".

Ten scale Pythagorean tuning

The screenshot shows a presentation slide titled "End of „normal” keys" in a LibreOffice Impress window. The slide contains four green boxes with white text, each representing a Pythagorean tuning ratio. The ratios are arranged from left to right: 1, 2197/2048, 16/13, 169/128, 13/8, 28561/16384, and 2. The text "End of „normal” keys" is centered below the ratios. The presentation interface includes a menu bar, a toolbar, a slide navigation pane on the left, and a properties pane on the right. The status bar at the bottom indicates the slide number (110 of 123) and the current slide is a blank slide.

1	$\frac{2197}{2048}$	$\frac{16}{13}$	$\frac{169}{128}$	$\frac{13}{8}$	$\frac{28561}{16384}$	2
---	---------------------	-----------------	-------------------	----------------	-----------------------	---

End of „normal” keys

Ten scale Pythagorean tuning

The slide displays the following mathematical relationships:

1	$\frac{2197}{2048}$	$\frac{16}{13}$	$\frac{169}{128}$	$\frac{13}{8}$	$\frac{28561}{16384}$	2
---	---------------------	-----------------	-------------------	----------------	-----------------------	---

Arrows indicate the derivation of the number $\frac{13}{8}$ from the other values:

- An arrow from $\frac{169}{128}$ to $\frac{13}{8}$ is labeled $\times \frac{13}{8}$.
- An arrow from $\frac{28561}{16384}$ to $\frac{13}{8}$ is labeled $\div 2$.

The slide is shown within a LibreOffice Impress window. The interface includes a menu bar (File, Edit, View, Insert, Format, Slide, Slide Show, Tools, Window, Help), a toolbar, a slide navigation pane on the left, a Properties pane on the right, and a status bar at the bottom.

Ten scale Pythagorean tuning

The slide displays a sequence of fractions representing Pythagorean tuning intervals. The fractions are arranged in a sequence from left to right, with arrows indicating multiplication and division operations between them.

The fractions shown are:

- 1
- $\frac{2197}{2048}$
- $\frac{16}{13}$
- $\frac{169}{128}$
- $\frac{371293}{262144}$
- $\frac{13}{8}$
- $\frac{28561}{16384}$
- 2

Arrows indicate the following operations:

- An arrow labeled $\times \frac{13}{8}$ points from the fraction $\frac{169}{128}$ to the fraction $\frac{371293}{262144}$.
- An arrow labeled $\div 2$ points from the fraction $\frac{371293}{262144}$ to the fraction $\frac{13}{8}$.

The interface shows the LibreOffice Impress application window with the title "Rass_v2 (1).pptx - LibreOffice Impress". The status bar at the bottom indicates "Slide 112 of 123 (12%)", "Blank Slide", and coordinates "0.82 / 4.47" and "0.00 x 0.00". The system tray shows the date and time: "Mon Jul 16, 19:33:14".

Ten scale Pythagorean tuning

The screenshot shows a presentation slide titled "Blank Slide" in LibreOffice Impress. The slide contains five musical intervals represented as fractions of strings, arranged from left to right:

- Interval 1: $\frac{2197}{2048}$
- Interval 2: $\frac{16}{13}$ and $\frac{169}{128}$
- Interval 3: $\frac{371293}{262144}$
- Interval 4: $\frac{13}{8}$ and $\frac{28561}{16384}$
- Interval 5: 2

The slide is displayed in a window titled "Rass_v2 (1).pptx - LibreOffice Impress". The interface includes a menu bar (File, Edit, View, Insert, Format, Slide, Slide Show, Tools, Window, Help), a toolbar, a "Slides" panel on the left, a "Properties" panel on the right, and a "Layouts" panel. The status bar at the bottom shows "Slide 113 of 123 (12%)", "Blank Slide", and coordinates "0.82 / 4.47" and "0.00 x 0.00".

Ten scale Pythagorean tuning

The slide displays a sequence of numbers and fractions illustrating the Pythagorean tuning process. The numbers are arranged in a sequence from left to right: 1, 2197/2048, 16/13, 169/128, 371293/262144, 13/8, 28561/16384, and 2. A curved arrow points from the fraction 16/13 to the fraction 13/8. Above the arrow is the multiplication symbol \times and the fraction 16/13. The slide is shown in a LibreOffice Impress window with a sidebar on the left showing slide thumbnails and a sidebar on the right showing properties and layouts.

1	$\frac{2197}{2048}$	$\frac{16}{13}$	$\frac{169}{128}$	$\frac{371293}{262144}$	$\frac{13}{8}$	$\frac{28561}{16384}$	2
---	---------------------	-----------------	-------------------	-------------------------	----------------	-----------------------	---

$\times \frac{16}{13}$

Ten scale Pythagorean tuning

The slide displays the following content:

$\times \frac{16}{13}$

1	$\frac{2197}{2048}$	$\frac{16}{13}$	$\frac{169}{128}$	$\frac{371293}{262144}$	$\frac{256}{169}$	$\frac{13}{8}$	$\frac{28561}{16384}$	2
---	---------------------	-----------------	-------------------	-------------------------	-------------------	----------------	-----------------------	---

The screenshot also shows the LibreOffice Impress interface with a sidebar of slide thumbnails (slides 104-115), a top menu bar, and a right-hand properties panel.

Ten scale Pythagorean tuning

The screenshot shows a presentation slide titled "Ten scale Pythagorean tuning" in LibreOffice Impress. The slide contains two main sections, labeled 1 and 2, each with a table of musical intervals. Section 1 is a 2x2 grid of intervals, and Section 2 is a 1x6 grid of intervals. The intervals are represented as fractions of frequencies.

1	$\frac{2197}{2048}$	$\frac{16}{13}$	$\frac{169}{128}$	$\frac{371293}{262144}$	$\frac{256}{169}$	$\frac{13}{8}$	$\frac{28561}{16384}$	2
---	---------------------	-----------------	-------------------	-------------------------	-------------------	----------------	-----------------------	---

The slide is displayed in a window titled "Rass_v2 (1).pptx - LibreOffice Impress". The interface includes a menu bar (File, Edit, View, Insert, Format, Slide, Slide Show, Tools, Window, Help), a toolbar, a Slides panel on the left, a Properties panel on the right, and a Layouts panel at the bottom right. The status bar at the bottom indicates "Slide 116 of 123 (12%)", "Blank Slide", and "2024/02/16 19:33:41".

Ten scale Pythagorean tuning

Slide 117 of 123 (12%)

1	$\frac{2197}{2048}$	$\frac{16}{13}$	$\frac{169}{128}$	$\frac{371293}{262144}$	$\frac{256}{169}$	$\frac{13}{8}$	$\frac{28561}{16384}$	2
---	---------------------	-----------------	-------------------	-------------------------	-------------------	----------------	-----------------------	---

$\times \frac{16}{13}$

Blank Slide

0.82 / 4.47 0.00 x 0.00

Polub

19%

Ten scale Pythagorean tuning

Slide 118 of 123 (128)

Blank Slide

0.82 / 4.47 0.00 x 0.00

Polish

Slide 118 of 123 (128)

11:44

Applications Places System

Rass_v2 (1).pptx - LibreOffice Impress

File Edit View Insert Format Slide Slide Show Tools Window Help

Format: Widescreen
Orientation: Landscape
Background: None
Master Slide: Blank Slide
Master Background
Master Objects
Master View

Layouts

Slide

1 $\frac{2197}{2048}$ $\frac{16}{13}$ $\frac{169}{128}$ $\frac{371293}{262144}$ $\frac{256}{169}$ $\frac{13}{8}$ $\frac{28561}{16384}$ $\frac{4096}{2197}$ 2

$\times \frac{16}{13}$

Ten scale Pythagorean tuning

The screenshot shows a LibreOffice Impress presentation slide titled "Blank Slide". The slide content consists of a sequence of numbers and fractions arranged in a row, representing a ten-scale Pythagorean tuning. The numbers are: 1, 2197/2048, 16/13, 169/128, 371293/262144, 256/169, 13/8, 28561/16384, 4096/2197, and 2. The fractions are displayed as stacked text. The slide is part of a presentation with 119 slides, and the current slide is 119 of 123 (12%). The status bar at the bottom shows the coordinates 0.82 / 4.47 and 0.00 x 0.00. The system tray at the bottom right shows the date and time: Mon Jul 10, 19:34:00.

1	$\frac{2197}{2048}$	$\frac{16}{13}$	$\frac{169}{128}$	$\frac{371293}{262144}$	$\frac{256}{169}$	$\frac{13}{8}$	$\frac{28561}{16384}$	$\frac{4096}{2197}$	2
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Ten scale Pythagorean tuning

Slide content:

1	$\frac{2197}{2048}$	$\frac{16}{13}$	$\frac{169}{128}$	$\frac{371293}{262144}$	$\frac{256}{169}$	$\frac{13}{8}$	$\frac{28561}{16384}$	$\frac{4096}{2197}$	2
---	---------------------	-----------------	-------------------	-------------------------	-------------------	----------------	-----------------------	---------------------	---

Annotations:

- Arrow from $\frac{16}{13}$ to $\frac{4096}{2197}$ labeled $\times \frac{16}{13}$
- Arrow from $\frac{4096}{2197}$ to $\frac{16}{13}$ labeled $\div 2$

Properties Panel:

- Slide: Format: Widescreen, Orientation: Landscape, Background: None, Master Slide: Blank Slide, Master Background, Master Objects, Master View
- Layouts: [Grid of layout thumbnails]

System Bar: Slide 120 of 120 (12%), 0.82 / 4.47, 0.00 x 0.00, Polub, 20°C, 21°C, 21°C, 24°C, 11°C, Mon Jul 10, 19:34:00

Ten scale Pythagorean tuning

Slide 121 of 123 (12%)

1	$\frac{2197}{2048}$	$\frac{32768}{28561}$	$\frac{16}{13}$	$\frac{169}{128}$	$\frac{371293}{262144}$	$\frac{256}{169}$	$\frac{13}{8}$	$\frac{28561}{16384}$	$\frac{4096}{2197}$	2

$\times \frac{16}{13}$

$\div 2$

Blank Slide

0.82 / 4.47 0.00 x 0.00

Polish

154%

Ten scale Pythagorean tuning

The screenshot shows a presentation slide titled "Ten scale Pythagorean tuning" displayed in LibreOffice Impress. The slide content is as follows:

1	$\frac{2197}{2048}$	$\frac{32768}{28561}$	$\frac{16}{13}$	$\frac{169}{128}$	$\frac{371293}{262144}$	$\frac{256}{169}$	$\frac{13}{8}$	$\frac{28561}{16384}$	$\frac{4096}{2197}$	2
---	---------------------	-----------------------	-----------------	-------------------	-------------------------	-------------------	----------------	-----------------------	---------------------	---

The interface includes a menu bar (File, Edit, View, Insert, Format, Slide, Slide Show, Tools, Window, Help), a toolbar, a slide navigation pane on the left, a Properties panel on the right, and a status bar at the bottom.

Ten scale Pythagorean tuning

Final 10-step scale

1	$\frac{2197}{2048}$	$\frac{32768}{28561}$	$\frac{16}{13}$	$\frac{169}{128}$	$\frac{371293}{262144}$	$\frac{256}{169}$	$\frac{13}{8}$	$\frac{28561}{16384}$	$\frac{4096}{2197}$	2
---	---------------------	-----------------------	-----------------	-------------------	-------------------------	-------------------	----------------	-----------------------	---------------------	---

Ten scale Pythagorean tuning

The screenshot shows a presentation slide titled "Final 10-step scale" and "Pythagoras 12-step scale". The slide is displayed in a LibreOffice Impress window. The "Final 10-step scale" is represented by a horizontal row of 10 boxes, each containing a fraction. The boxes are colored in a sequence of dark green, light green, and dark green. The "Pythagoras 12-step scale" is represented by two rows of boxes. The top row has 6 boxes, and the bottom row has 12 boxes. The boxes are colored in a sequence of light green, dark green, and grey.

Final 10-step scale

1	$\frac{2197}{2048}$	$\frac{32768}{28561}$	$\frac{16}{13}$	$\frac{169}{128}$	$\frac{371293}{262144}$	$\frac{256}{169}$	$\frac{13}{8}$	$\frac{28561}{16384}$	$\frac{4096}{2197}$	2
---	---------------------	-----------------------	-----------------	-------------------	-------------------------	-------------------	----------------	-----------------------	---------------------	---

Pythagoras 12-step scale

	$\frac{256}{243}$	$\frac{32}{27}$		$\frac{729}{519}$	$\frac{128}{81}$	$\frac{16}{9}$						
1	$\frac{9}{8}$	$\frac{81}{64}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{27}{16}$	$\frac{243}{128}$						2

- Consider all possible musical systems with t keys in an octave.
- As far as the **problem of transposing melodies** from one key to another is concerned, the **best** among these systems is the t -system in which all the **adjacent keys are apart the same interval**.
- Let this interval be given by a number r .
- We start with our first key, in the first octave, which has frequency value 1.
- The second key has frequency value $1 \times r = r$, , the third key, $r \times r = r^2$, the fourth $r^2 \times r = r^3$, and so on, until the t^{th} key, which will have frequency value $r^{t-2} \times r = r^{t-1}$.
- Since the scale has t keys, the $(t + 1)^{\text{th}}$ key **starts a new octave**, so that the value of this key is on one hand $r^{t-1} \times r = r^t$, but on the other hand is 2.

- We thus have $r^t = 2$, or r is the t^{th} **root of 2**.
- Thus, the musical intervals in such equally distanced scale are equal to

$$r = (2)^{\frac{1}{t}}.$$

- A musical scale with t equal intervals $r = (2)^{\frac{1}{t}}$ is called **equally tempered**, or **equal temperament**.

Equal temperament for $t = 12$

The screenshot shows a presentation slide in LibreOffice Impress. The slide title is "Equal temperament". Below the title is a diagram illustrating the division of an octave. A horizontal line is marked with 12 equal intervals, starting at 1 and ending at 2. Above the line, there are 12 arrows pointing to the right, each labeled with $\times r$. Below the line, the equation $r \times r \times \dots \times r = r^{12} \Rightarrow r = \sqrt[12]{2}$ is displayed.

Equal temperament

$\times r$ $\times r$ $\times r$ $\times r$ $\times r$ $\times r$ $\times r$ $\times r$ $\times r$ $\times r$ $\times r$ $\times r$

1 2

$r \times r \times \dots \times r = r^{12} \Rightarrow r = \sqrt[12]{2}$

- The **decahonic piano of Leszek Mozdzer**, which physical implementation will be presented during the **concert at Nowa Miodowa Hall on Thursday, 13th July, at 19:00**, uses **equally tempered 10 scale musical system**.
- Mathematically the system has a remarkable property that the **differences in musical intervals** between **each of its ten keys** and the corresponding keys of the **Pythagorean $f = 13, t = 10, k = 7$ system**, are about **an order of magnitude smaller**, than the corresponding distances differences between keys of the **equally tempered 12 piano scale** and the keys of its **$f = 3, t = 12, k = 7$ Pythagorean counterpart**.