Decaphonic piano
Its mathematics, physics and sound

Aleksander Bogucki

Instytut Fizyki Doswiadczalnej UW

Pawel Nurowski

Centrum Fizyki Teoretyczne PAN & Guangdong Technion - Israel Intitute of Technology

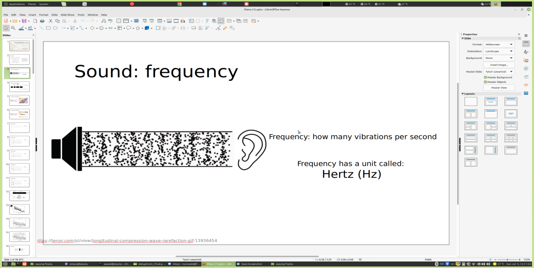
GRIEG meets Chopin
Warszawa, July 11, 2023

- I am a mathematician from the Center for Theoretical Physics of PAS, and a coordinator of the grant **SCREAM** (Symetry, Curvature Reductions, and EquivAlence Methods) from the **National Science Center of Poland**, in the **GRIEG** grant scheme, founded from the **Norwegian Financial Mechanism 2014-2021** (with project registration number 2019/34/H/ST1/00636.)
- Related to this grant, there is a conference **GRIEG meets Chopin** being held now in Warsaw, and I tried to make a concert for the conference participants. I had an idea of employing a **pianist**, who has both **Chopin** and **Grieg** in her repertoir to play the concert, but I failed.

- Then I thought that in **jazz**, one can superpose several compositions in one piece. Through my friends I got a contact to a jazz pianist I admire, namely **Leszek Mozdzer**, and to my surprise he agreed to give a concert for our conference, provided that I will help him to built a piano he always dreamed about. Actually he needed a mathematical help for this built, and has chosen **the first mathematician who ever contacted him.**
- It turned out that the mathematics he needed, was not demanding at all, but then I realised that an actual implementation of the frequencies which Leszek needs in his **piano** suffers from lack of solutions to a number of **physical** and **technological** problems.

- So I went to my Alma Mater, the **Physics Department of University of Warsaw**, and asked for a PhD student who is very talented in both **experimental** physics and **piano music**.
- It is how I encountered **Aleksander Bogucki** without whom this project would never even had been started.
- Aleksander introduced me to **Andrzej Włodarczyk**, a world expert in the **historic piano restoration** - privately Aleksander's longterm **piano tuner**. Mr Włodarczyk joined our team with enthusiasm. He **revealed many of his piano building secrets** for us. This elevated our project to the professional level. Again, without Andrzej Włodarczyk's contribution to the project, we would not be here today.

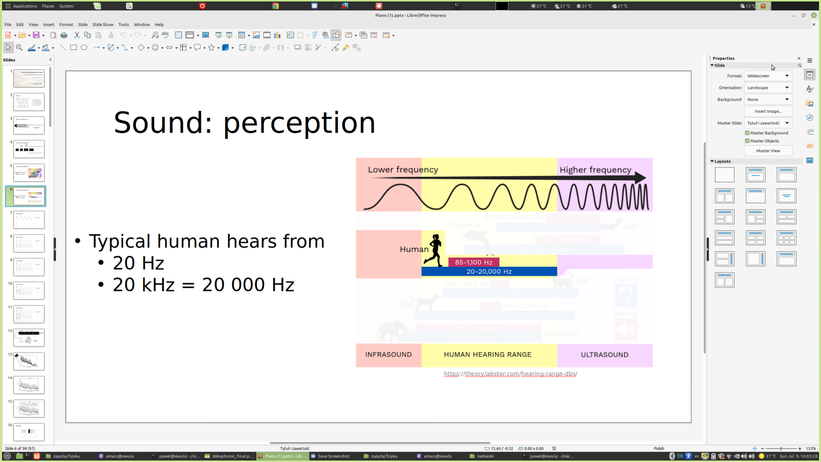
Part I: MATHEMATICS



- Sound is a **vibration of air preasure** that **we sense with our ears**[1].
- **The rate** at which these vibration heat our eardrums is called **the frequency of the sound.** This is **measured in Hertz**s, which is the **number of vibrations per second**.

[1] Non decaphonic part of this talk uses a lot from the youtube video of **Yuval Nov**. One can consult his video at https://www.youtube.com/watch?v=nK2jYk37Rlg&ab_channel=Formant in case one is lost in this part

Frequency range



- The frequency of musical tones is in the range of, say **50 Hertz**, to **few thousends Hertz**.

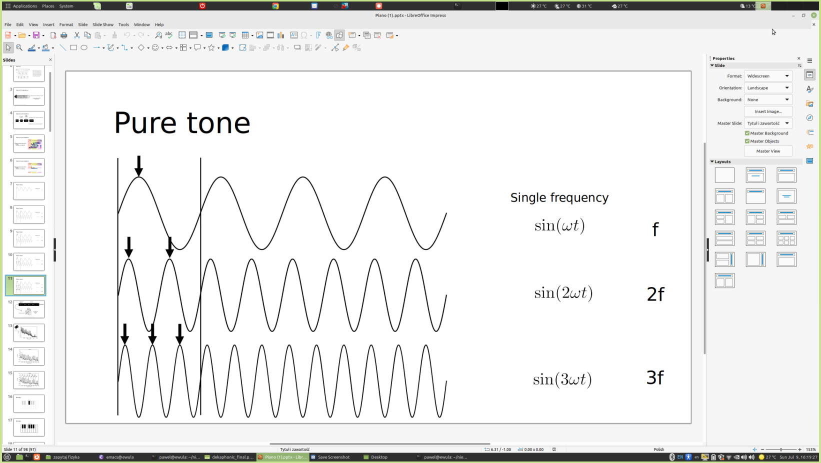
High frequency $\Longleftrightarrow$ high pitch

- Our brain percives higher frequencies detected by our ears as higher pitch
- ▸ Link

Pure tones

- All sounds are made, in a specific way, by building blocks, namely by the **pure tones**.
- We model a pure tone with sound frequency f, by a function *sinus*, which changes with time t as $\sin(2\pi f\,t)$.

Sines of pure tones with various frequencies

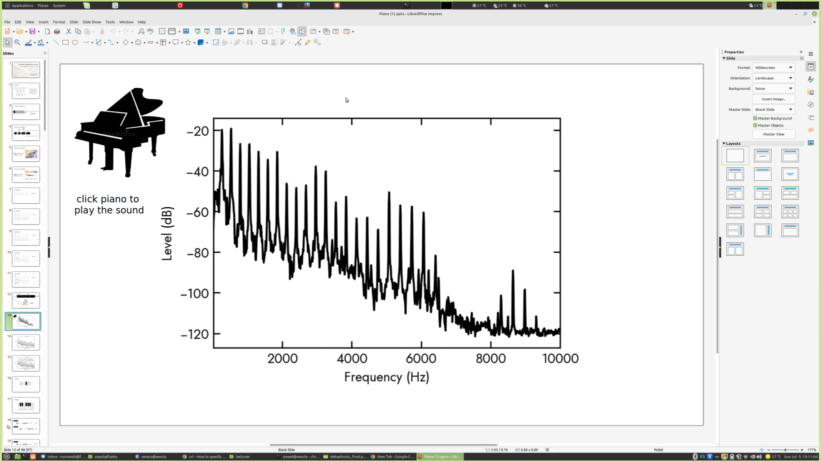


- Relation between ω and f is $\omega = 2\pi f$.

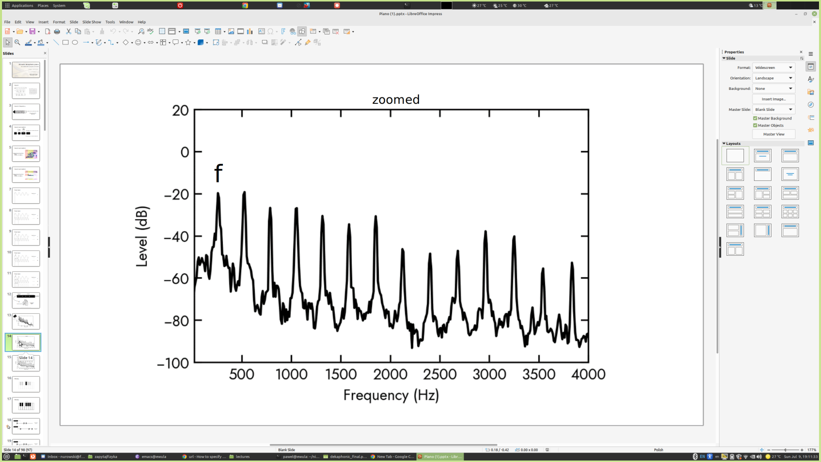
- When we **force a musical instrument to play a pure tone with frequency** f, an (actual infinite) **number of other pure tones, with frequencies** $2f$, $3f$, $4f$, **etc**, called **overtones, is created.**

Link

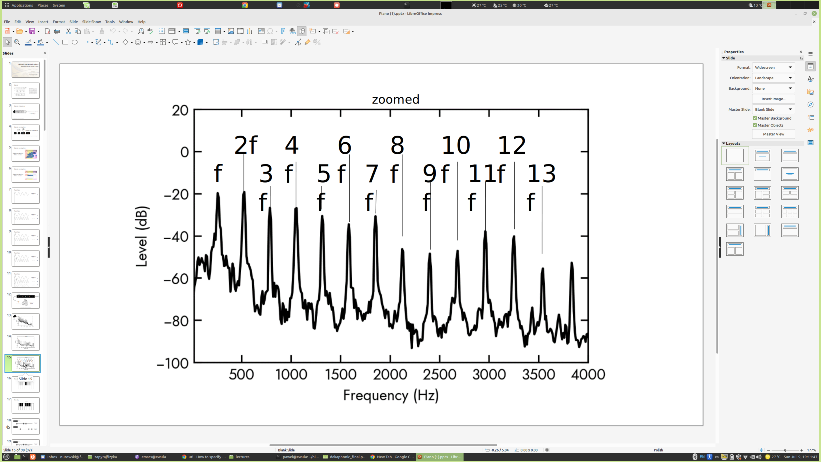
Tones and overtones for a piano



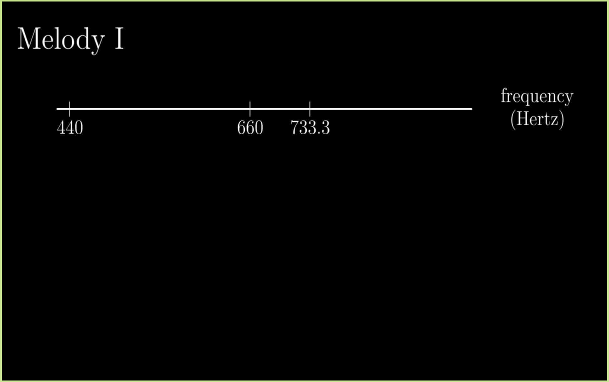
Tones and overtones for a piano



Tones and overtones for a piano

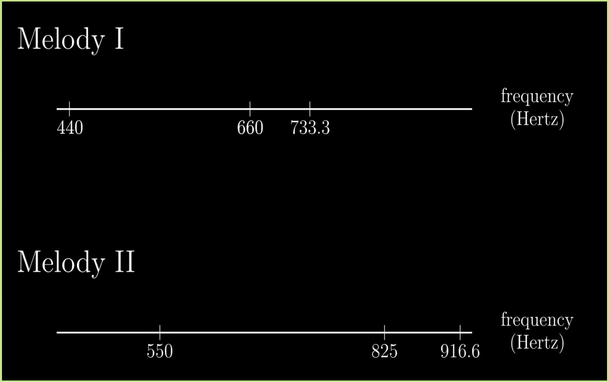


- For a melody, we need more than one tone.



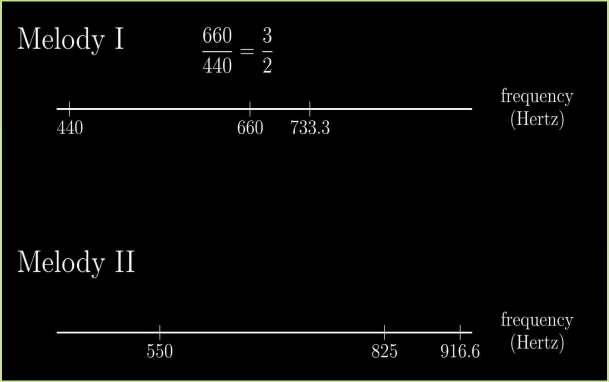
Two, or one melody?

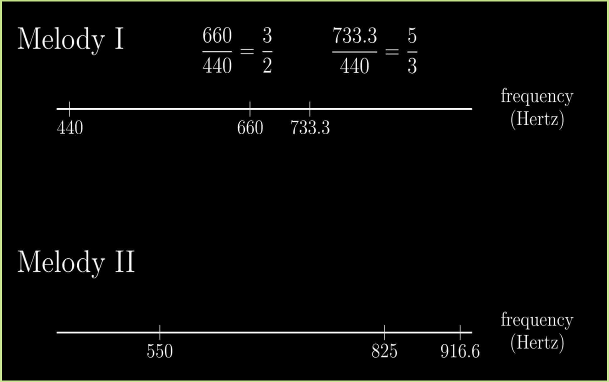
- Link
- What makes these into melodies? Answer: the change in pitch.
- Are these melodies the same? Strictly speaking NO, since none of the notes of the first melody is the same as a note of the second melody.
- But we strongly feel that these melodies are the same.



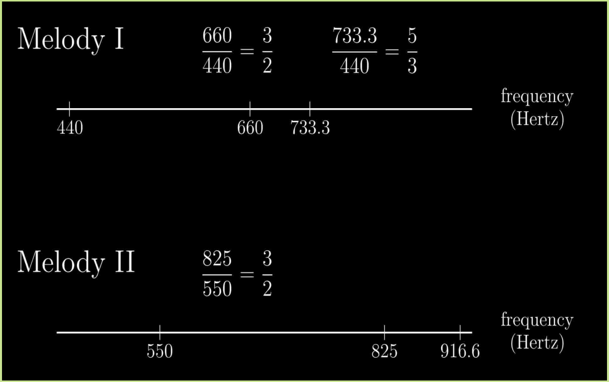
- Different frequencies, but melodies are the same! Why?

What is a melody?

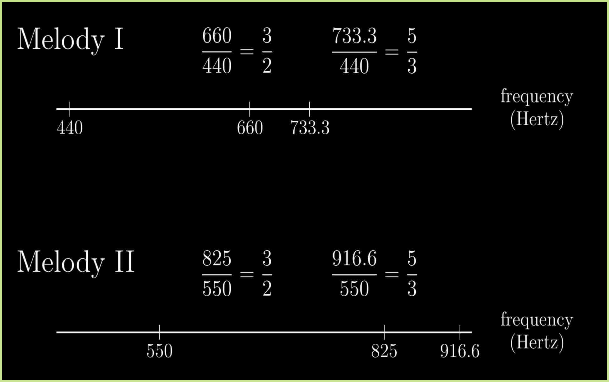



Melody I
660/440 = 3/2
733.3/440 = 5/3
440
660
733.3
frequency (Hertz)
Melody II
550
825
916.6
frequency (Hertz)

What is a melody?



What is a melody?

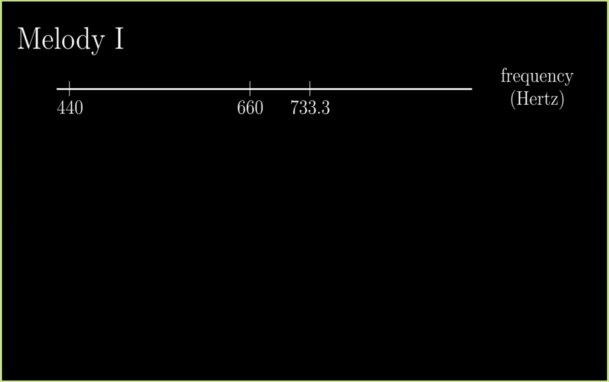


What is a melody?


Melody I
$\frac{660}{440} = \frac{3}{2}$
$\frac{733.3}{440} = \frac{5}{3}$
440
660
733.3
frequency
(Hertz)
Melody II
$\frac{825}{550} = \frac{3}{2}$
$\frac{916.6}{550} = \frac{5}{3}$
550
825
916.6
frequency
(Hertz)

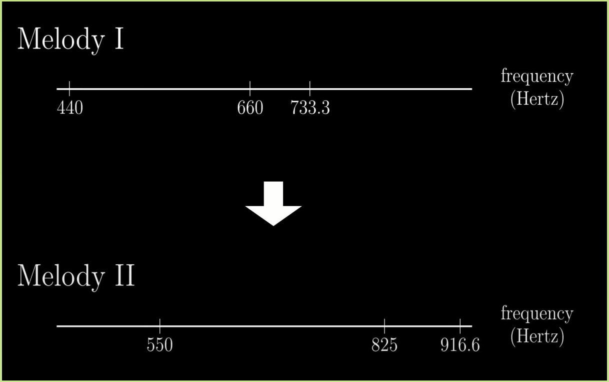
- These are the **ratios of neighbouring frequences** which **are the same!**
- Conclusion:

the same melodies $\Longleftrightarrow$ **same ratios between the keys**

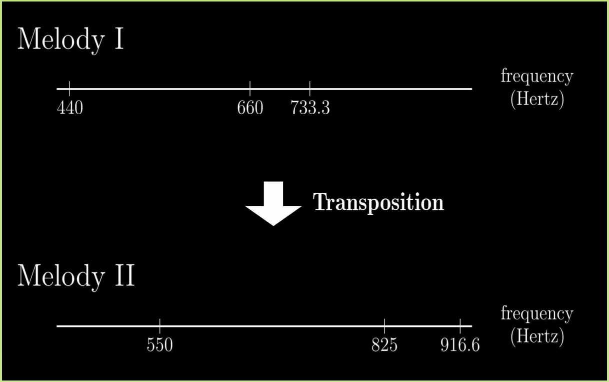


The act of changing of all the frequencies of a given melody by the same factor - so that the ratios between the frequencies stay the same - is called **transposition.**

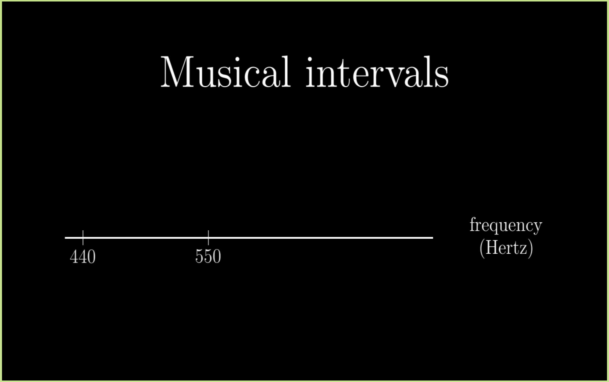
The act of changing of all the frequencies of a given melody by the same factor - so that the ratios between the frequencies stay the same - is called **transposition.**



The act of changing of all the frequencies of a given melody by the same factor - so that the ratios between the frequencies stay the same - is called **transposition.**

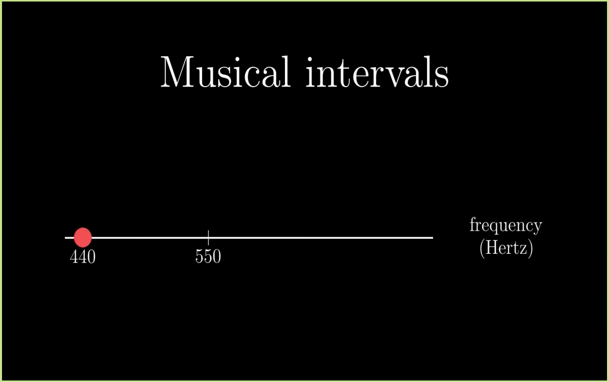


The act of changing of all the frequencies of a given melody by the same factor - so that the ratios between the frequencies stay the same - is called **transposition.**



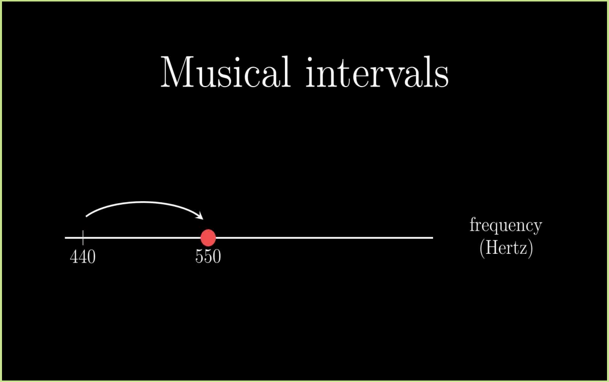
- Between any two tones there is a **musical interval**

Musical intervals



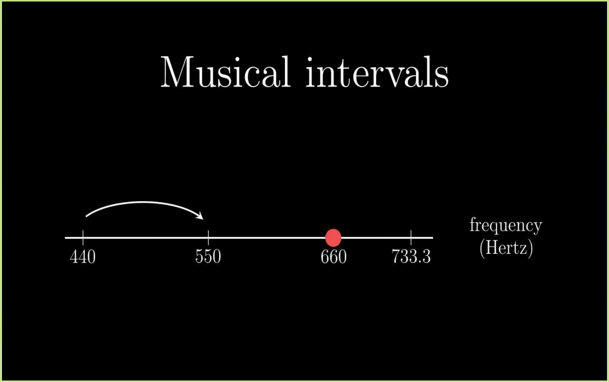
- musical interval = musical distance between two tones

Musical intervals

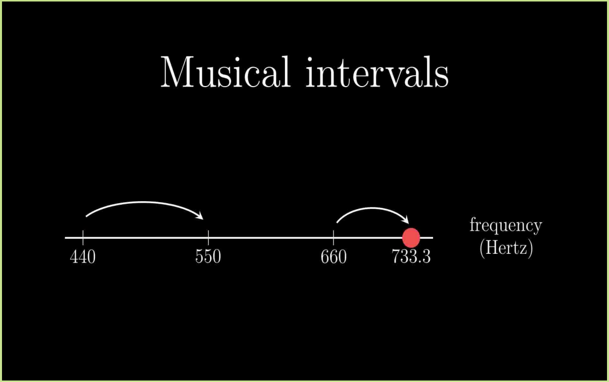


- musical interval = musical distance between two tones

Musical intervals

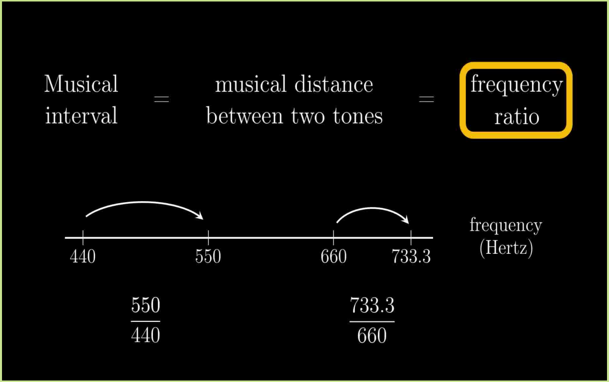


- musical interval = musical distance between two tones



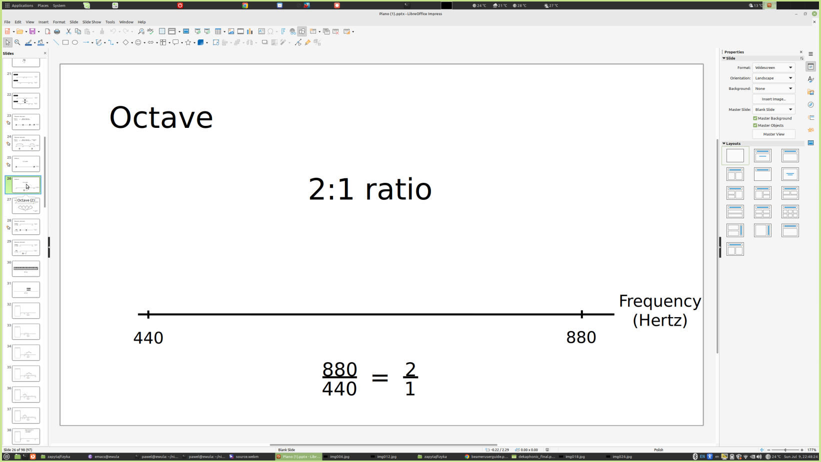
- musical interval = musical distance between two tones

Musical intervals



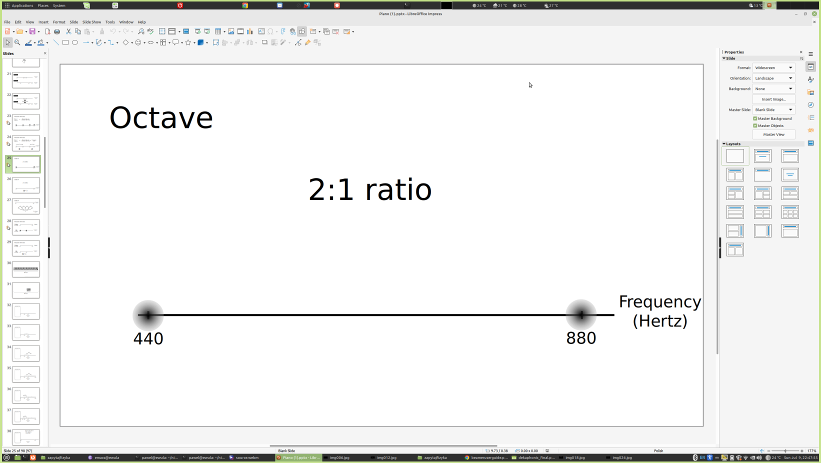
- We now introduce two mportant intervals.

The octave



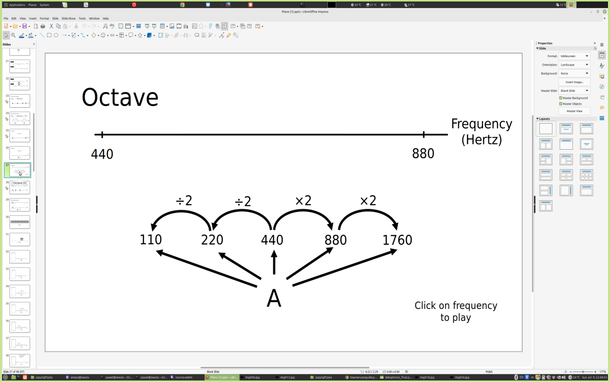
- **The octave** = musical interval with **ratio 2:1**

The octave



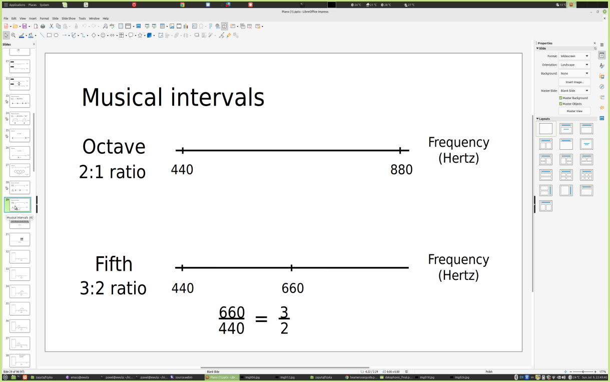
- The **octave is very pleasent** to the ear.
- Two tones, an **octave apart**, played together sound **very harmonious**

The octave equivalence



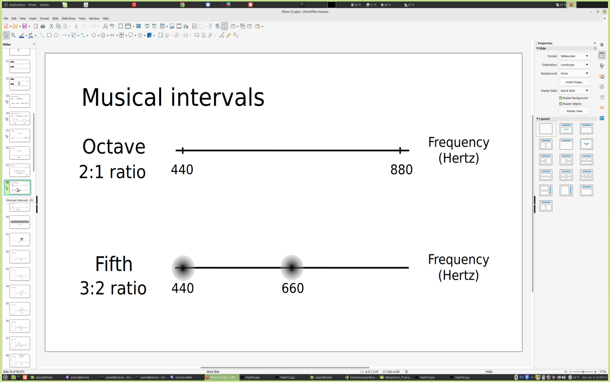
- They are **highly similar**; they are so similar that musicians denote them by **the same letter**, and musicologists say that they belong to the same **pitch class**.
- They are considered to be musically **equivalent**.
- We say about the **octave equivalence**.

The fifth



- Another important **musical interval** is given by the **ratio 3:2**.
- It is called **the fifth**.

The fifth



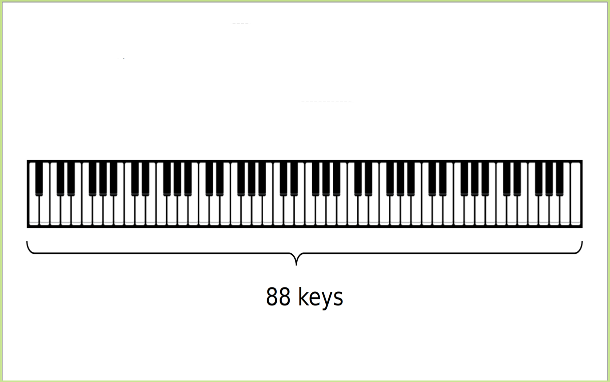
- Another important **musical interval** is given by the **ratio 3:2**.
- It is called **the fifth.** It is also very pleasent to the ear.

- Due to the **octave equivalence**, the **higher harmonics**

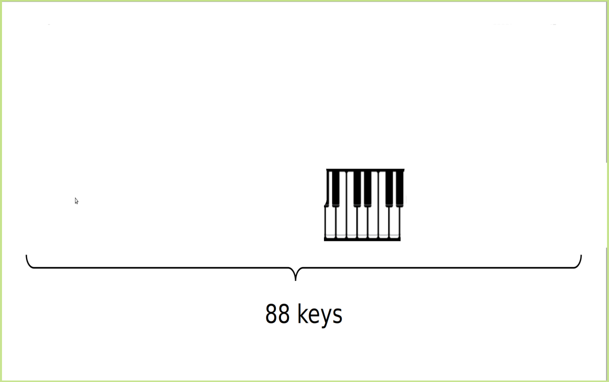
$$2f, 3f, 4f, 5f, 6f, 7f, 8f, 9f, 10f, 11f, 12f, 13f, \ldots,$$

 can be **placed between** f **and** $2f$.
- Since for melodies only the **ratios between the tones** are important, so f, $2f$, $4f$, ..., $2^k f$ are **octave equivalent**, as well as $\frac{1}{2}f$, $\frac{1}{4}f$, ... $\frac{1}{2^k}f$, for every integer k.
- For example the next musical interval betwen the harmonics after f and $2f$ is f and $3f$. But this, due to the octave equivalence is the same, as an interval between f and $\frac{3}{2}f$, which explains **pleasance of the fifth.** It is the **next** harmonious musical interval to consider, **after the octave.**

The piano keyboard

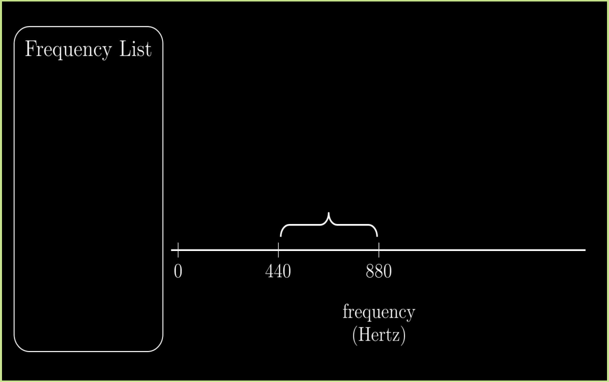


- We are in a position to chose pleasent sounds for the piano keys.
- The piano keyboard has $2 + \mathbf{12} \times \mathbf{7} + 2 = 88$ keys.

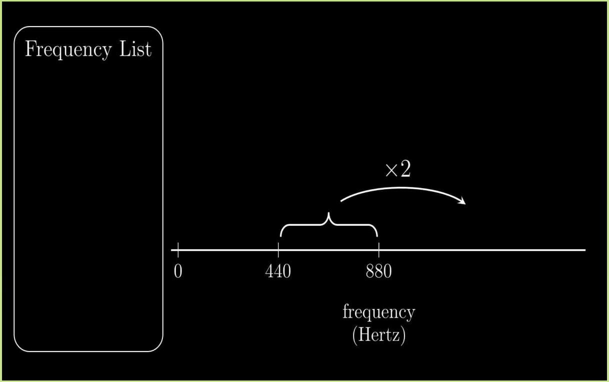


- Look at the above above patern of **twelve** keys. In the keyboard, the pattern is repeated **seven** times, an octave apart from each other. This gives $\mathbf{12} \times \mathbf{7} = \mathbf{84}$ keys. To these 84 keys, two additional keys are added on each of the lateral ends of the keyboard.

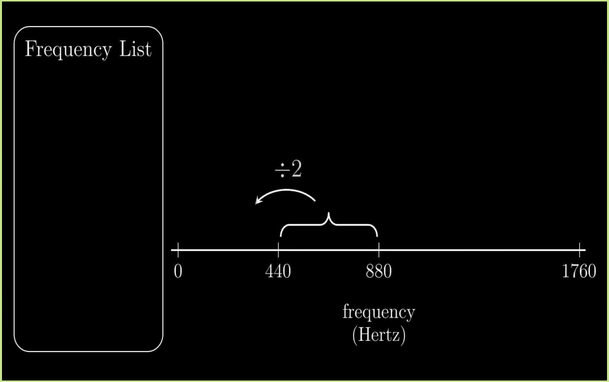
- Which tones are chosen for the **twelve** keys in an octave?



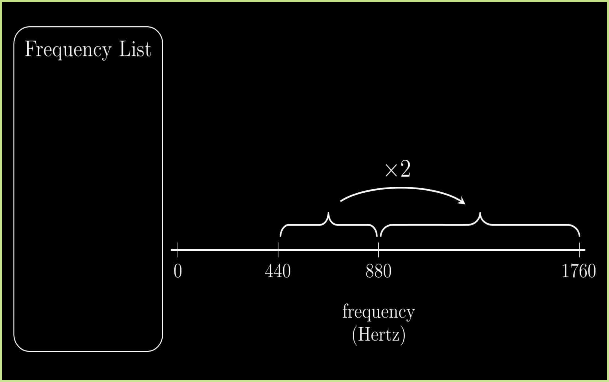
- We start with an octave, say 440Hz – 880Hz.
- We can restrict ourselves to chose frequencies in one octave only, because frequencies for the keys in other octaves will be obtained either



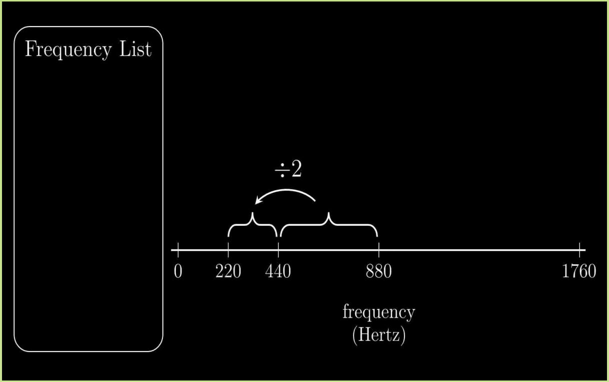
- by **multiplying** all the frequencies by 2, or



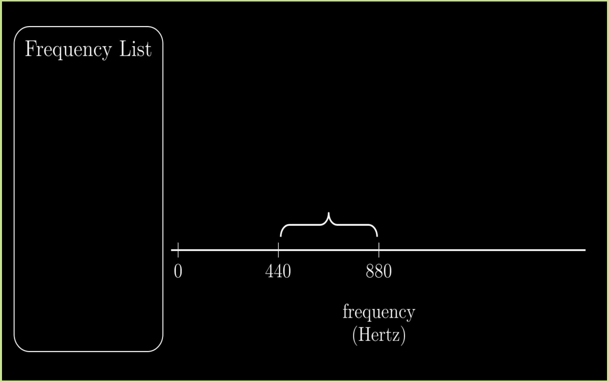
- by **dividing** frequencies by 2.



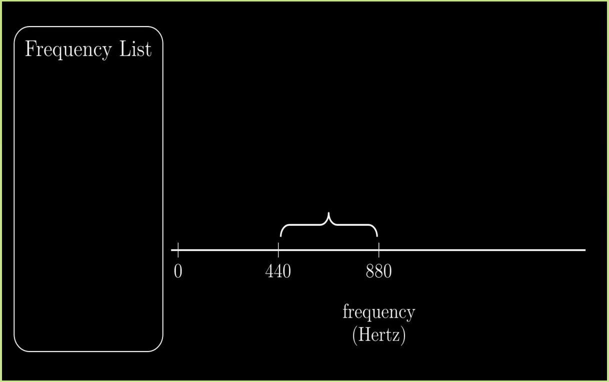
- For example, the **next octave** will have the frequency range: 880Hz–1760Hz,



- and the **previous octave** will have the frequency range: 220Hz–440Hz.

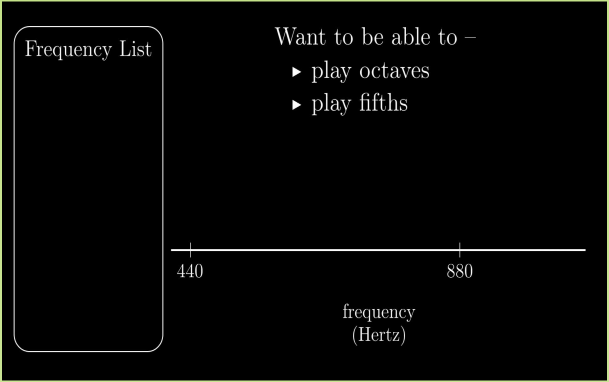


- We will now make frequency choices for the keys in our chosen octave 440Hz-880Hz.

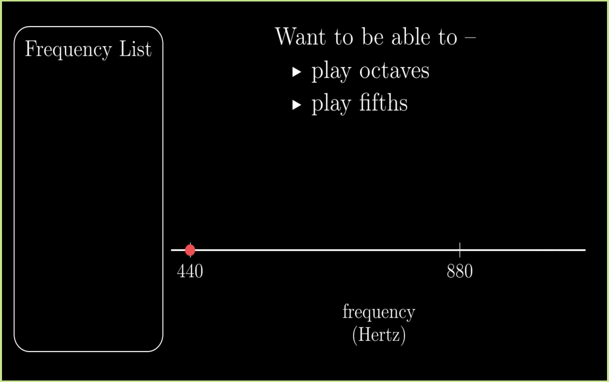


- Of course we **want to play octaves**, so we have to have two frequencies octave apart in our frequency list.
- We also **want to play fifths**, so the corresponding frequency should also be in our list.

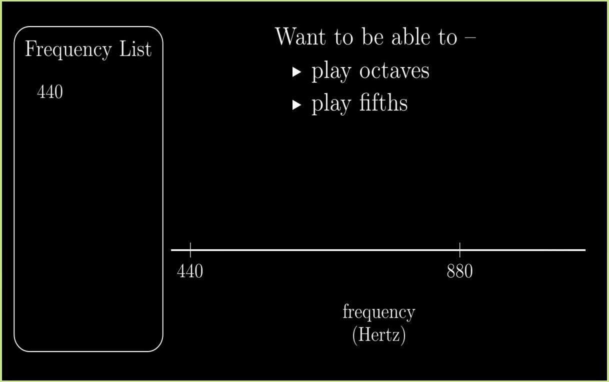
Here is as it goes:



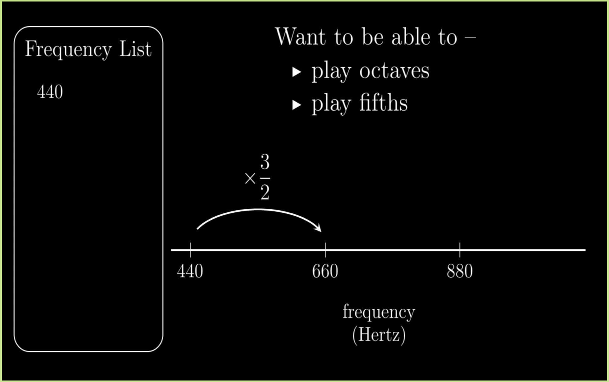
Here is as it goes:



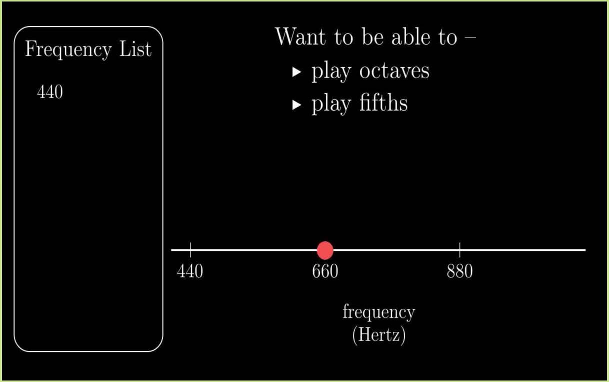
Here is as it goes:



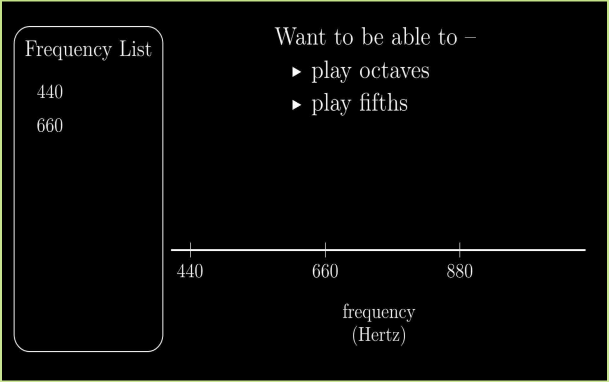
Here is as it goes:



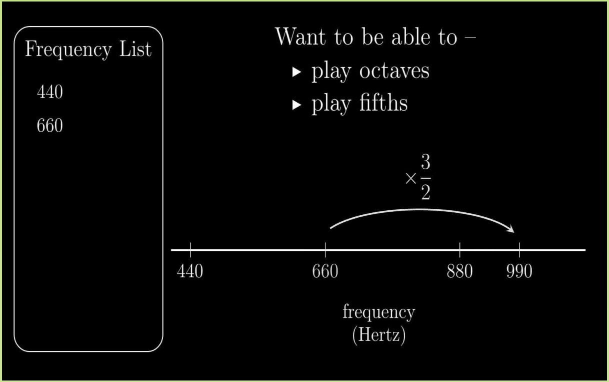
Here is as it goes:



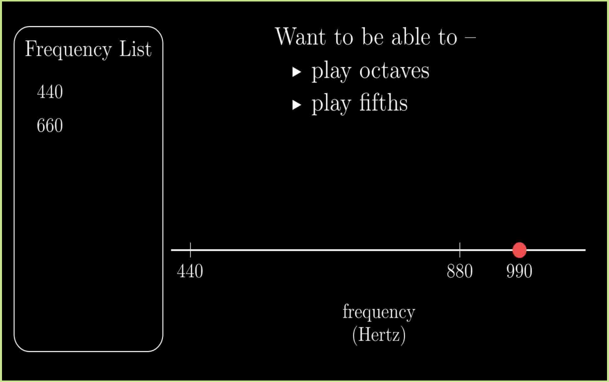
Here is as it goes:



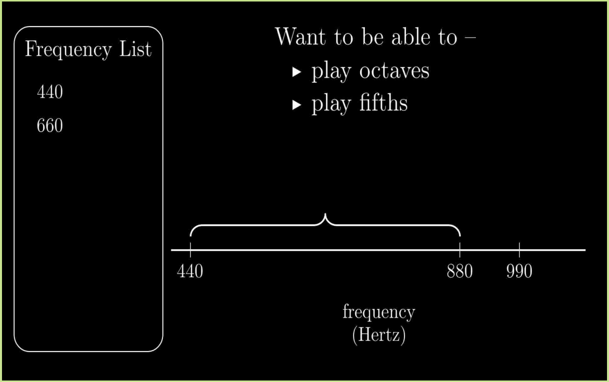
Here is as it goes:



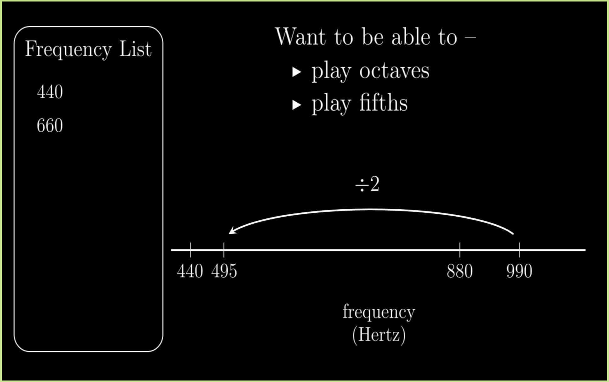
Here is as it goes:



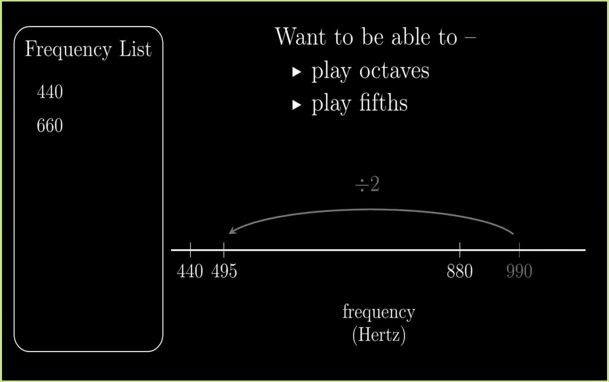
Here is as it goes:



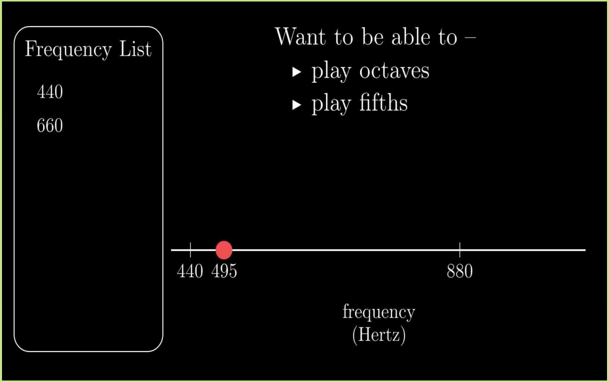
Here is as it goes:



Here is as it goes:



Here is as it goes:



Here is as it goes:

Frequency List

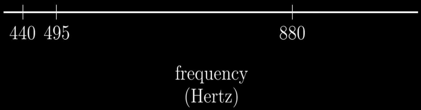
440

660

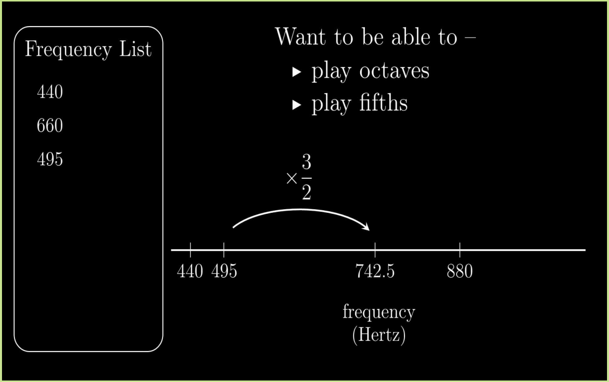
495

Want to be able to –

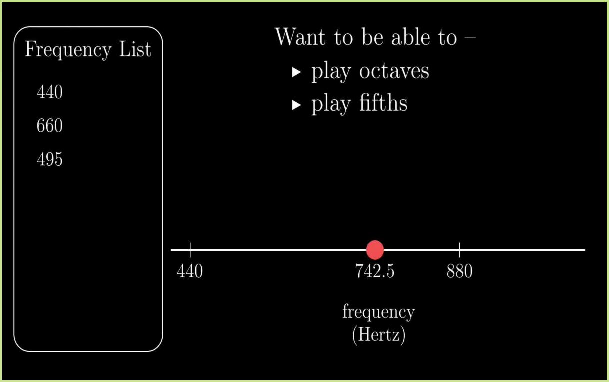
- play octaves
- play fifths



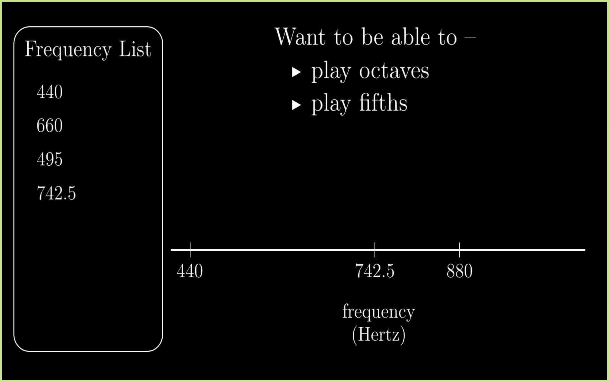
Here is as it goes:



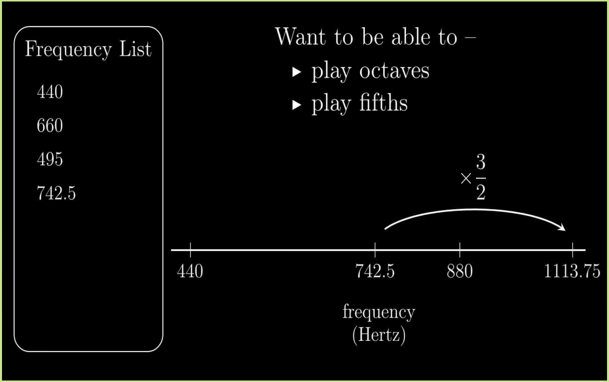
Here is as it goes:



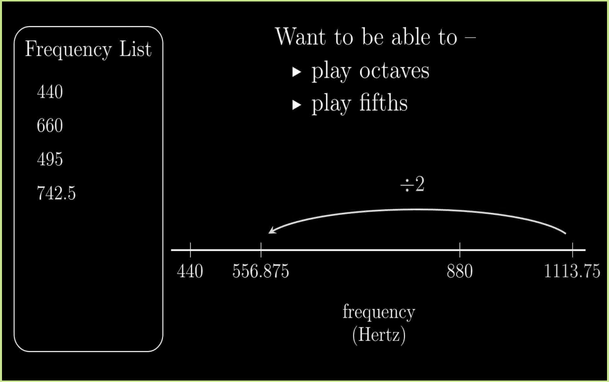
Here is as it goes:

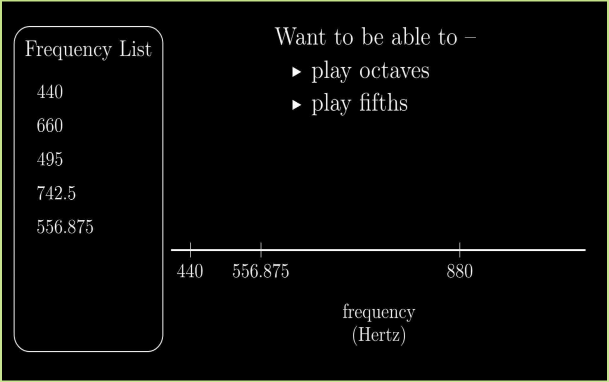


Here is as it goes:

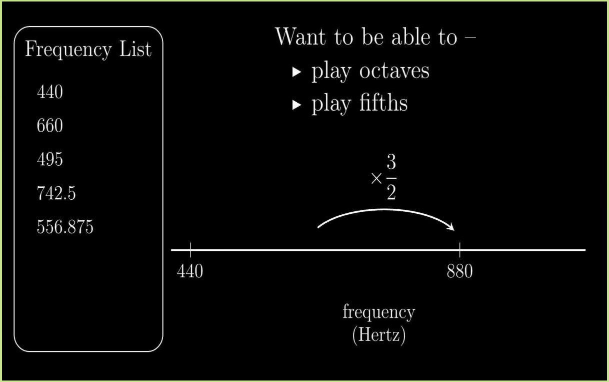


Here is as it goes:



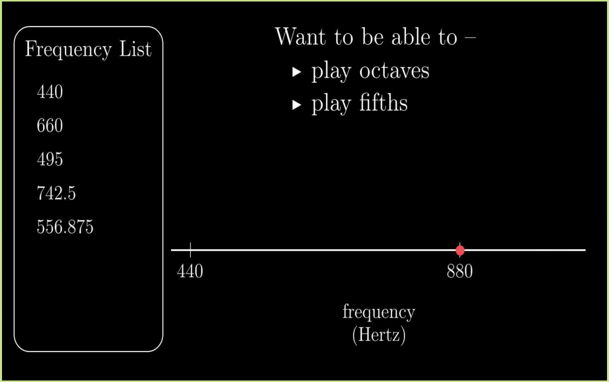


- When does this stop?

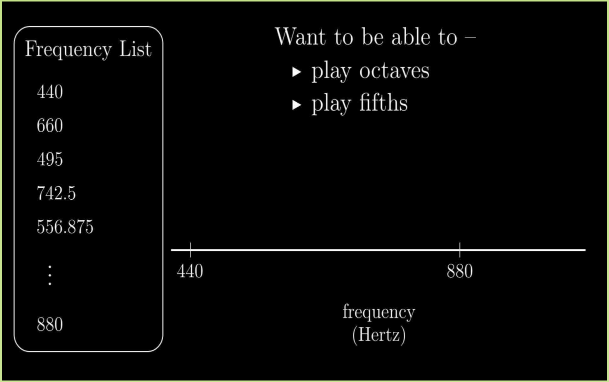


- Well... the best would be if we eventually arrived at 880Hz.

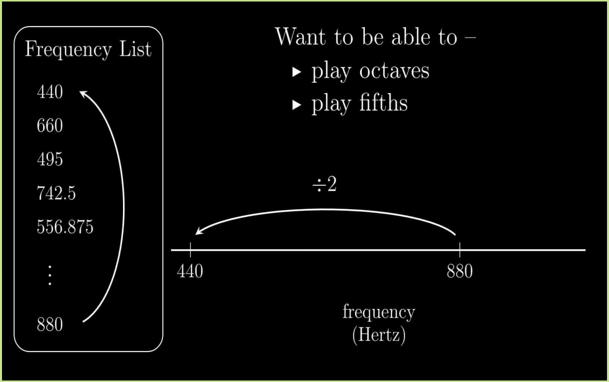
When does it stop?



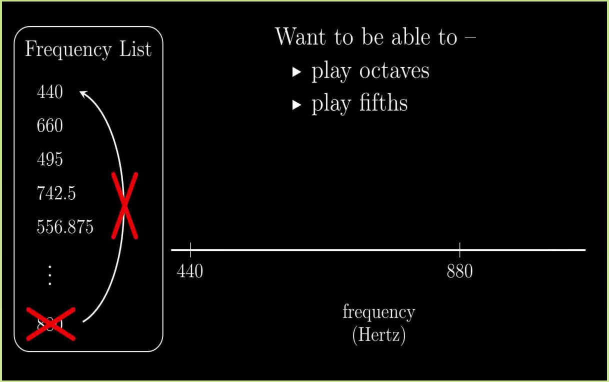
When does it stop?



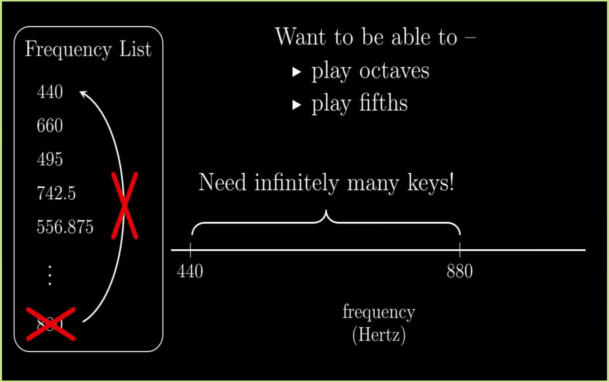
When does it stop?



It never stops!



It never stops!



If get back to 440 –

n = no. of times went up a fifth

k = no. of times went down an octave

$$440 \cdot \left(\frac{3}{2}\right)^n \cdot \left(\frac{1}{2}\right)^k = 440$$

If get back to 440 –

n = no. of times went up a fifth

k = no. of times went down an octave

$$\cancel{440} \cdot \left(\frac{3}{2}\right)^n \cdot \left(\frac{1}{2}\right)^k = \cancel{440}$$

$$3^n = 2^{n+k}$$

If get back to 440 –

n = no. of times went up a fifth

k = no. of times went down an octave

$$\cancel{440} \cdot \left(\frac{3}{2}\right)^n \cdot \left(\frac{1}{2}\right)^k = \cancel{440}$$

$$\text{odd number} = 3^n = 2^{n+k} = \text{even number}$$

contradiction

It never stops!

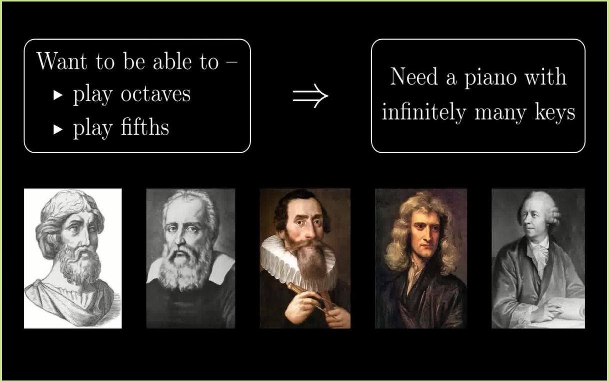
Want to be able to –

- play octaves
- play fifths

$\Rightarrow$

Need a piano with infinitely many keys

It never stops!



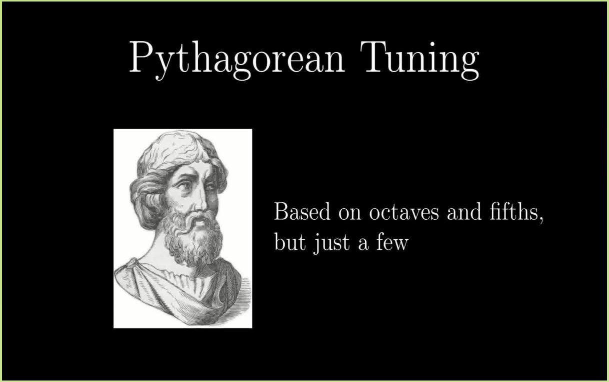
A choice of note frequencies for an octave –

“tuning”,

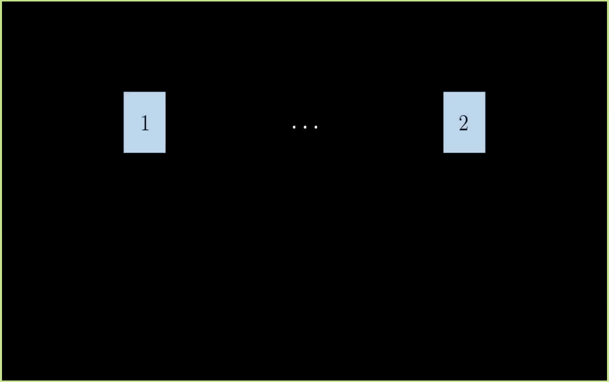
“intonation”,

“temperament”

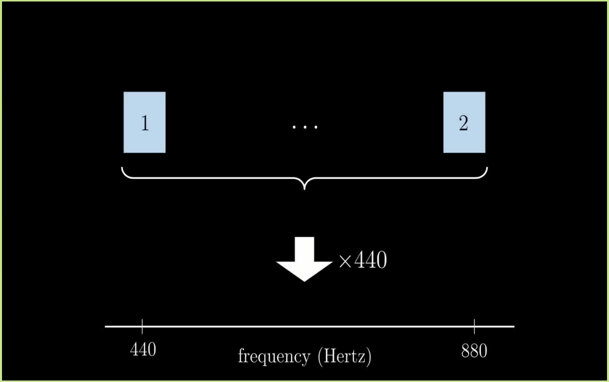
- There are numerous choices of tuning systems!

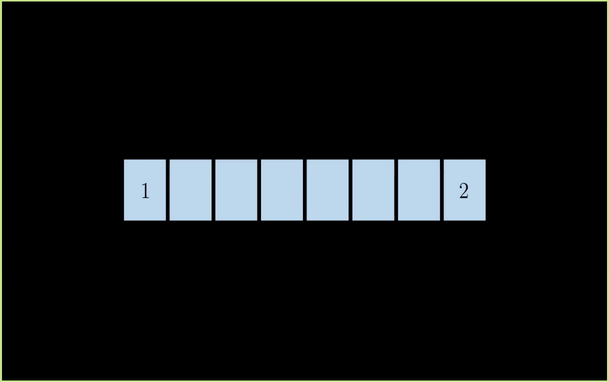

Pythagorean Tuning
Based on octaves and fifths,
but just a few

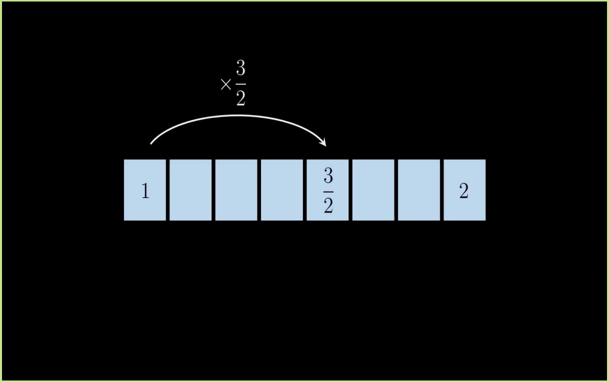
Pythagorian tuning



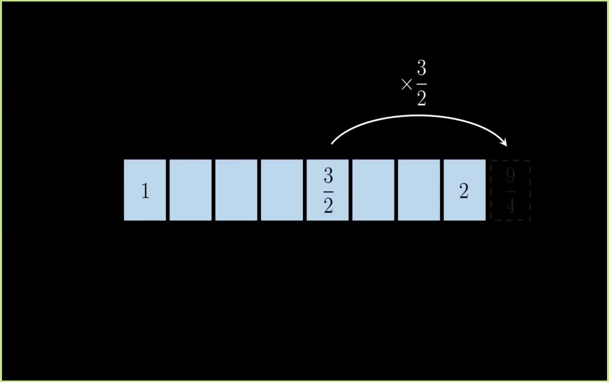
Pythagorian tuning



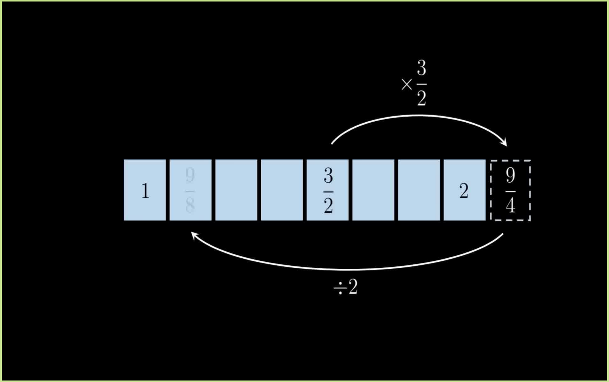

1
2

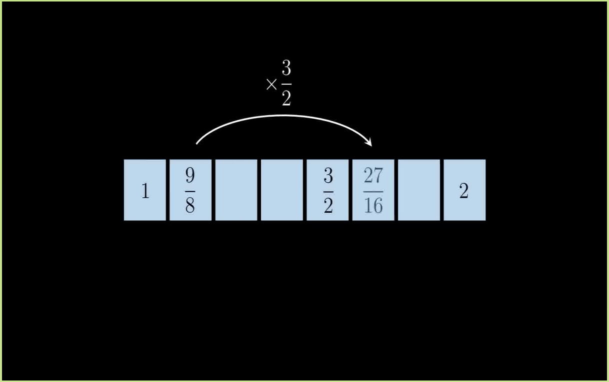

$\times \frac{3}{2}$
1
$\frac{3}{2}$
2

Pythagorian tuning

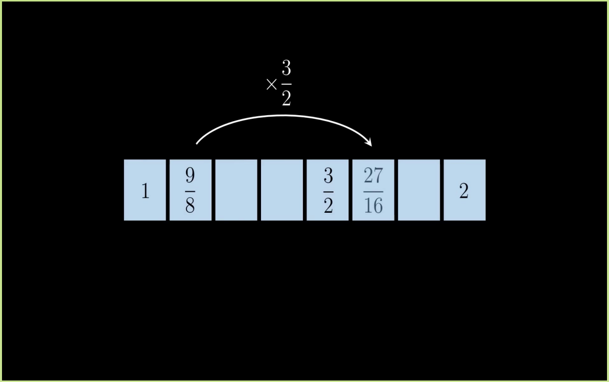


Pythagorian tuning

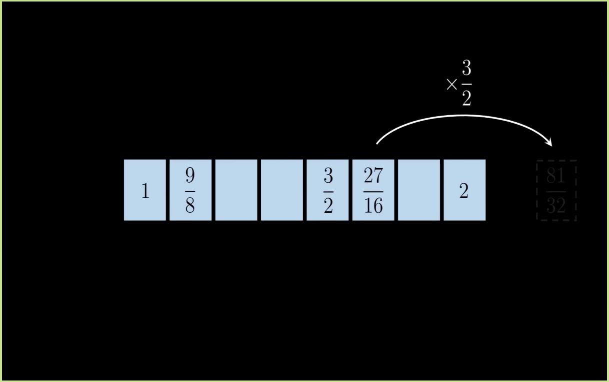



$\times \frac{3}{2}$
1
$\frac{9}{8}$
$\frac{3}{2}$
$\frac{27}{16}$
2

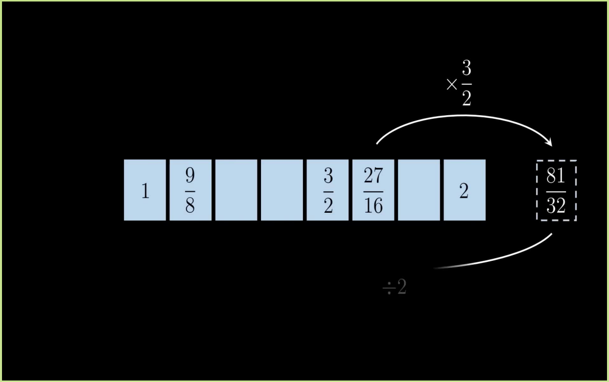
Pythagorian tuning

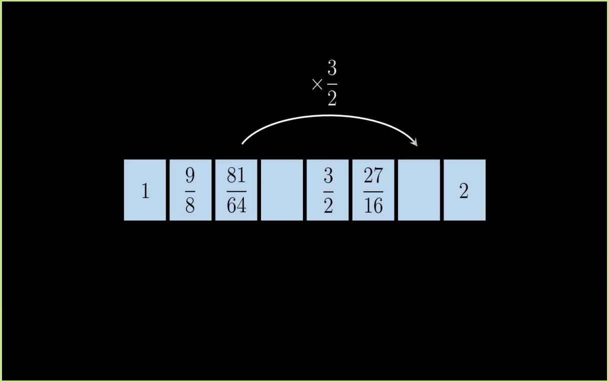


Pythagorian tuning

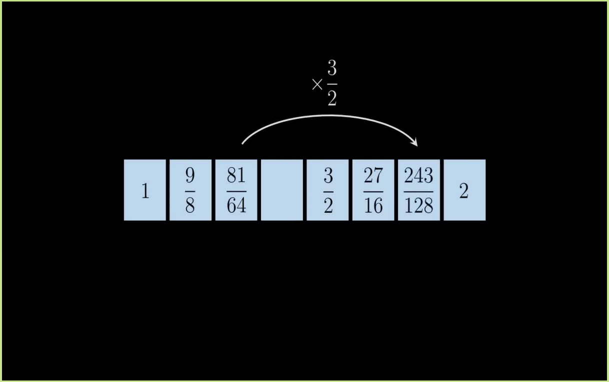


Pythagorian tuning



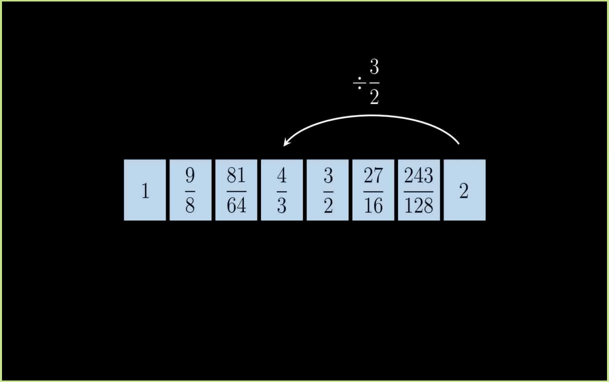

$\times \frac{3}{2}$
1
$\frac{9}{8}$
$\frac{81}{64}$
$\frac{3}{2}$
$\frac{27}{16}$
2

Pythagorian tuning



Pythagorian tuning

1	$\frac{9}{8}$	$\frac{81}{64}$		$\frac{3}{2}$	$\frac{27}{16}$	$\frac{243}{128}$	2



- Uff!...we assigned frequencies for the seven keys.
- How do they sound?

Pythagorian tuning: sounds of the seven white keys

Link

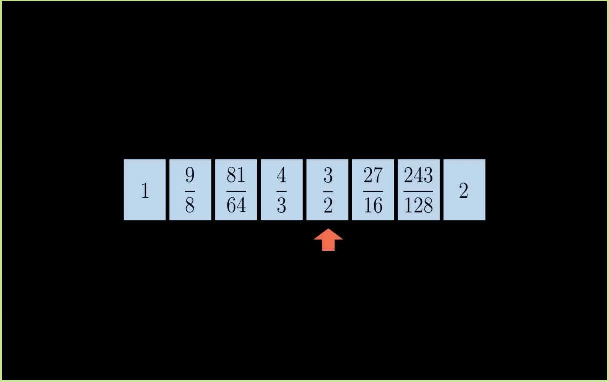
- Sounds familiar, ha?
- It is enough to play a simple melody.

Pythagorian tuning: simple melody

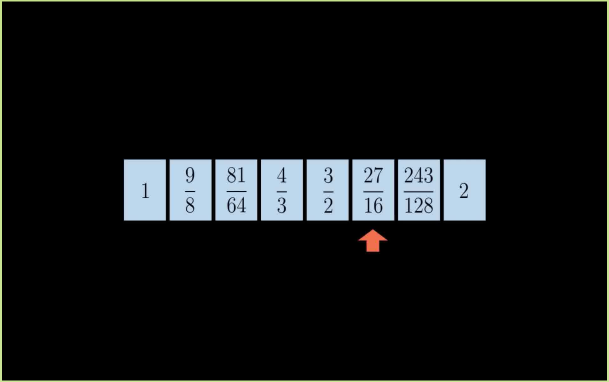
▸ Link

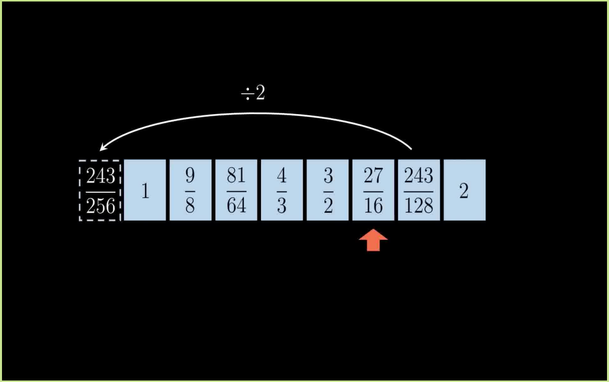
Pythagorian tuning: can we transpose?

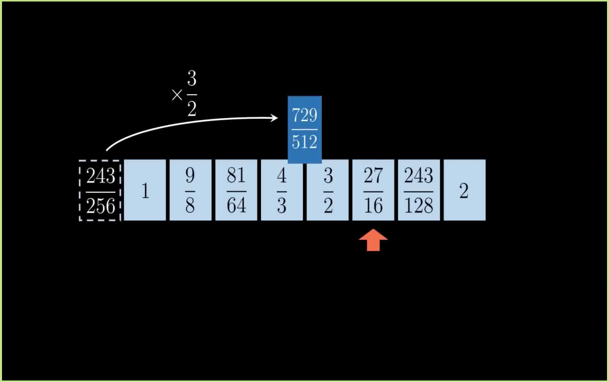
- There is however, a problem with a transposition: We started our melody from the fifth, i.e. $\frac{3}{2}$ key. If we started from the next key, $\frac{27}{16}$, we would be missing one note, to play the same melody :((.



- There is however, a problem with a transposition: We started our melody from the fifth, i.e. $\frac{3}{2}$ key. If we started from the next key, $\frac{27}{16}$, **we would be missing one note**, to play the same melody :((.

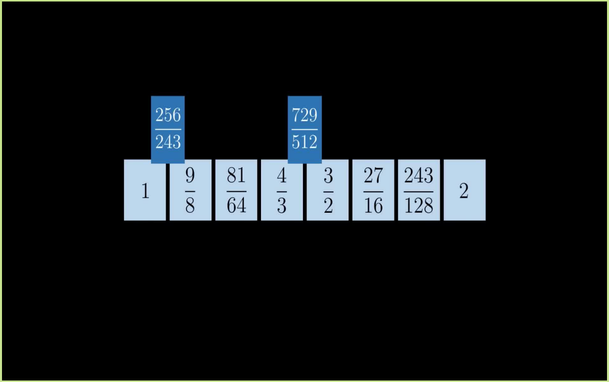



÷2
243/256
1
9/8
81/64
4/3
3/2
27/16
243/128
2

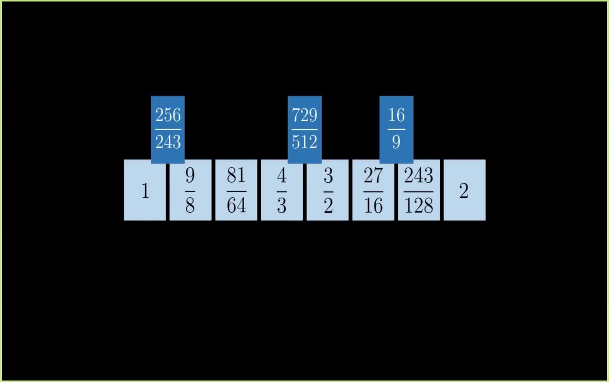


- We need to add only one key more, we again can play the same melody, now starting at $\frac{27}{16}$.
- Link

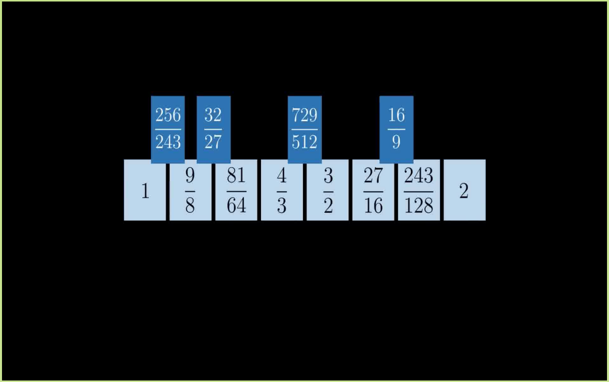
Pythagorian tuning: creating the black keys

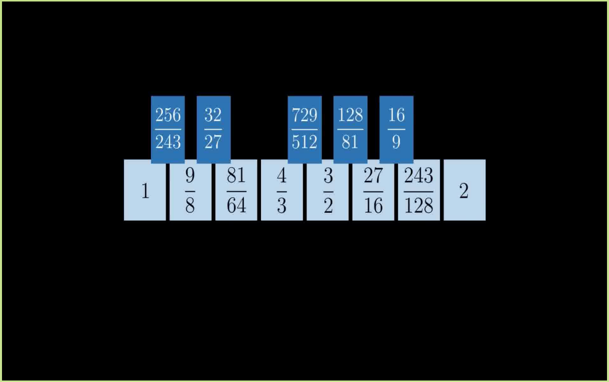


Pythagorian tuning: creating the black keys

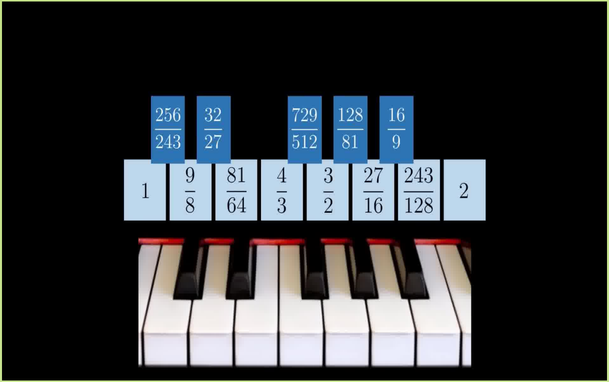


Pythagorian tuning: creating the black keys

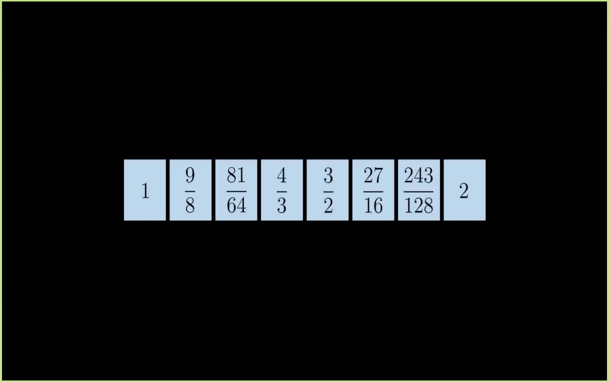




- Uff...We have 12 keys! 7 whites, and 5 blacks!

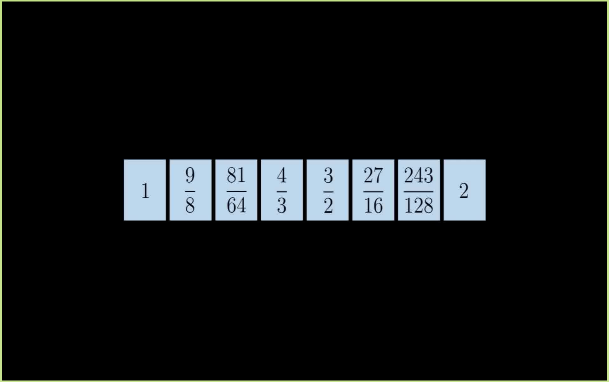


- What we have now is pretty much as is the fundamental pattern of the twelve keys of the piano keyboard.
- The Pythagorian system, has more trouble issues than just having troubles with transposition. The **just intonation** -yet another tuning system - is introduced to cure one of them.



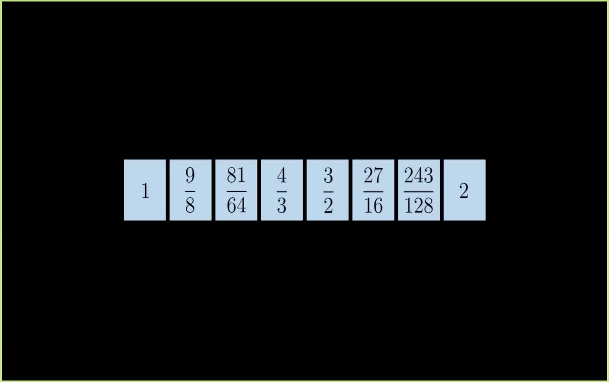
- One problem of the Pythagorian tuning is caused by the **beats.**
- Some important **chords** - the multiple harmoneous tones played at once - sound better in **just intonation.**

Just intonation: elimination of beats



- In particular, instantenous play of the **1st, 3rd and 5th note**, in the white keys, is an important chord, called **the major chord**.
- Link
- And the problem is with the **5th overtone** of the first note, and the **4th overtone** of the 3rd note.

Just intonation: elimination of beats



- It is as simple as:

$$4 \times \frac{81}{64} \simeq 5.0625 \simeq 5 \times 1.$$

- Because a superposition of two sinusoidal waves with so close frequencies produces this:
- Link

Just intonation

Just	1	$\frac{9}{8}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{15}{8}$	2
Pyth.	1	$\frac{9}{8}$	$\frac{81}{64}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{27}{16}$	$\frac{243}{128}$	2

$$\frac{81/64}{5/4} = 1.0125$$

Just intonation

Just	1	$\frac{9}{8}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{15}{8}$	2
Pyth.	1	$\frac{9}{8}$	$\frac{81}{64}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{27}{16}$	$\frac{243}{128}$	2

$$\frac{243/128}{15/8} = 1.0125$$

Just intonation

Just	1	$\frac{9}{8}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{15}{8}$	2
Pyth.	1	$\frac{9}{8}$	$\frac{81}{64}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{27}{16}$	$\frac{243}{128}$	2

$$\frac{27/16}{5/3} = 1.0125$$

Just intonation

Just	1	$\frac{9}{8}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{15}{8}$	2
Pyth.	1	$\frac{9}{8}$	$\frac{81}{64}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{27}{16}$	$\frac{243}{128}$	2

- Long time ago people observed that

$$\left(\frac{3}{2}\right)^{12} \simeq 2^7.$$

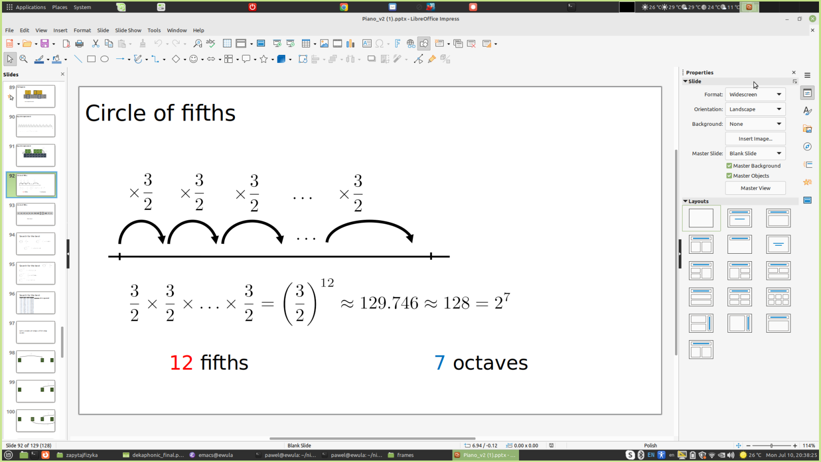
- This is the same as saying that

$$\left(\frac{3}{2}\right)^{\frac{12}{7}} \simeq 2.$$

- Well...

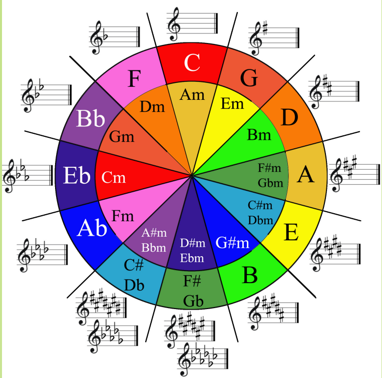
$$\left(\frac{3}{2}\right)^{\frac{12}{7}} - 2 \simeq 0.00387547.$$

Circle of fifths

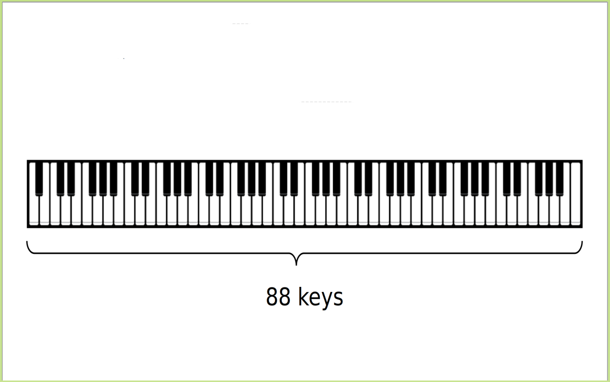


- This means that starting in the pitch class *C* and heating the successive key notes by the musical intervals of the fifth, the thirteenth heated fifth will sound as *C*.
- Well..., with a good aproximation.

Circle of fifths



- Link



- This, in particular explains $88 = 2 + \mathbf{12} \times \mathbf{7} + 2$ keys of the piano!
- **twelve** keys in each octaves, and **seven** octaves, to go around in terms of the fifths to the same pitch class C.

- For the pourpose of this lecture, I will call the **modulus of the difference** $\left|\left(\frac{3}{2}\right)^{\frac{12}{7}} - 2\right| = \mu_{(3,12,7)}$, a **comma**, of the tuning system.
- In the tuning systems we considered here, **the comma** $\mu_{(3,12,7)} = 0.00387547$ is related to the **three natural numbers** f, t, k which are:
  - $f = 3$, the third harmonic, which we used in the form of the ratio $\frac{3}{2}$, to generate our piano keys steps;
  - $t = 12$, the number of keys in one octave of our tuning system;
  - $k = 7$, the number of octaves needed to go from C back to C, by jumping through the keys by the intervals of $\frac{3}{2}$.

- **Question**: Can we have a Pythagorian tuning system with t keys in every octave, which has k octaves and which is generated from the first octave $1 \longleftrightarrow 2$ by a harmonic[2] f, such that
  - its comma $\mu_{(f,t,k)} < 0.01$,
  - the number of tones $t \times k < 100$,
  - its generating harmonic $f < 64$?
- I call this system Pythagorian, although it is based on f rather than $f = 3$. But this is a straightforward generalization.

[2] or a subharmonic $1/f$

Theorem

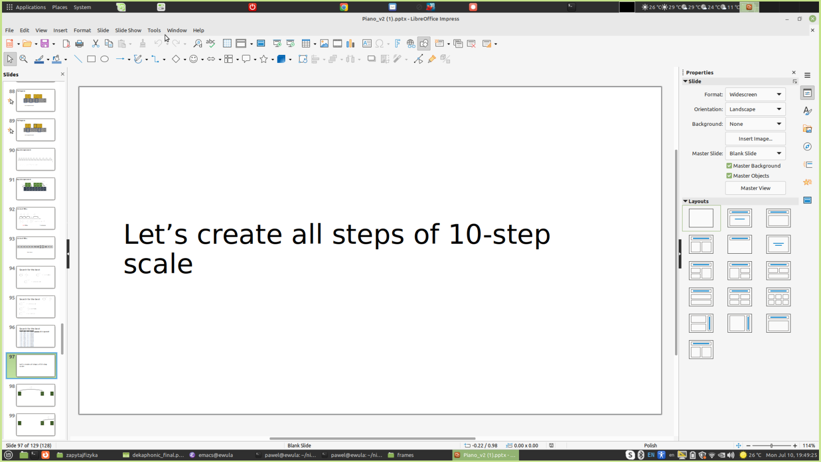
The only values of (f, t, k) that answer the question in positive are in the following table:

f	t	k	μ
$\frac{3}{2}$	12	7	0.00387547
$\frac{11}{8}$	13	6	0.00631819
$\frac{13}{8}$	10	7	0.000871016
$\frac{17}{16}$	23	2	0.00808825
$\frac{27}{16}$	4	3	0.00905446
$\frac{29}{16}$	7	6	0.00135602

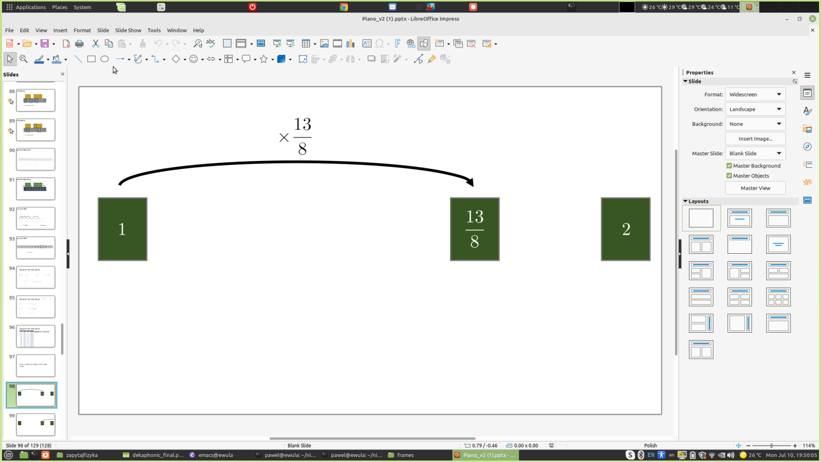
f	t	k	μ
$\frac{33}{32}$	45	2	0.00156911
$\frac{37}{32}$	19	4	0.0070528
$\frac{39}{32}$	7	2	0.00151358
$\frac{41}{32}$	14	5	0.00158879
$\frac{43}{32}$	7	3	0.00744747
$\frac{47}{32}$	9	5	0.00241079
$\frac{53}{32}$	11	8	0.00123505
$\frac{55}{32}$	9	7	0.00639457
$\frac{57}{32}$	6	5	0.000737349
$\frac{59}{32}$	9	8	0.00971721

- We see that among all Pythagorian systems, based on $f < 56$, the system with $f = 13$, $t = 10$, $k = 7$ is the system with the **smallest comma**.
- It is precisely the system which **Leszek Możdėr** wanted us to design for him. His motivation for using $t = 10$, $k = 7$ is kind of 'mystic'; totally unclear for us.
- The Pythagorian **decaphonic piano** has **six white** keys and **four black** keys.
- Its octave with **six white** keys is generated by $f = \frac{13}{8}$, which plays the role of the 'fifth', when compared to the $t = 12$, $k = 7$ Pythagorian system.
- Since we have **ten** keys, and $k = 7$, to cover a passage from C to C via the 'fifths' of $\frac{13}{8}$, the piano needs 70 keys, only.
- It should be stressed, that in our analysis we did not insist on having $t = 10$ keys! The **decimal/decaphonic** system, with **seven** octaves, was distinguished by pure mathematics.

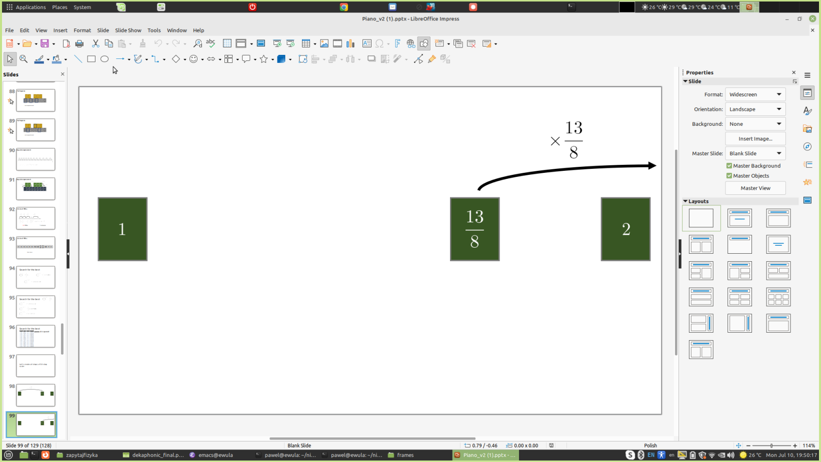
Ten scale Pythagorian tuning



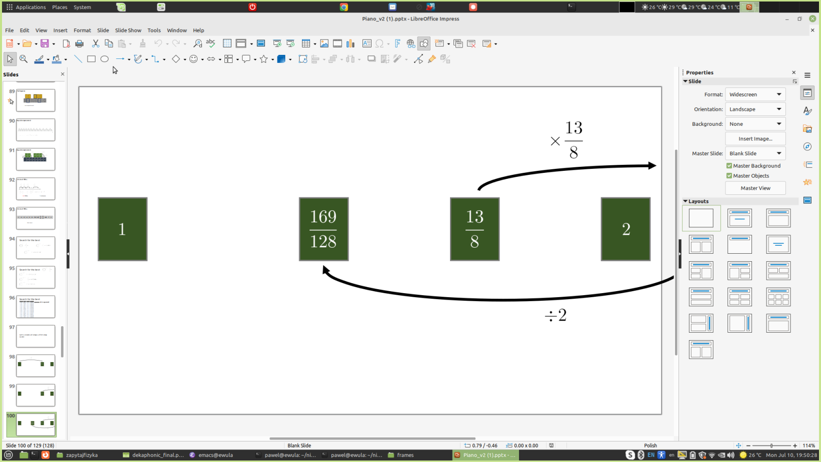
Ten scale Pythagorian tuning



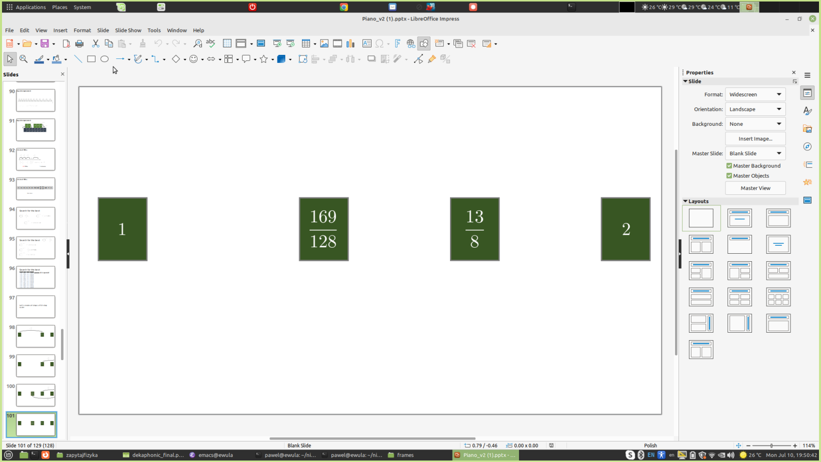
Ten scale Pythagorian tuning



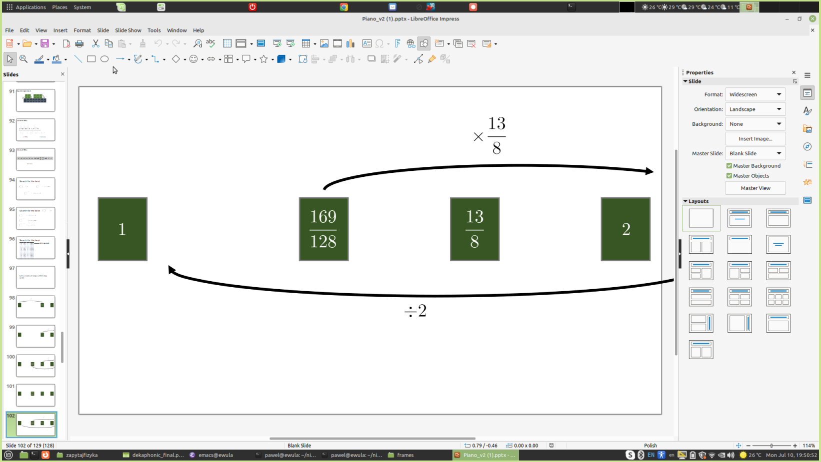
Ten scale Pythagorian tuning



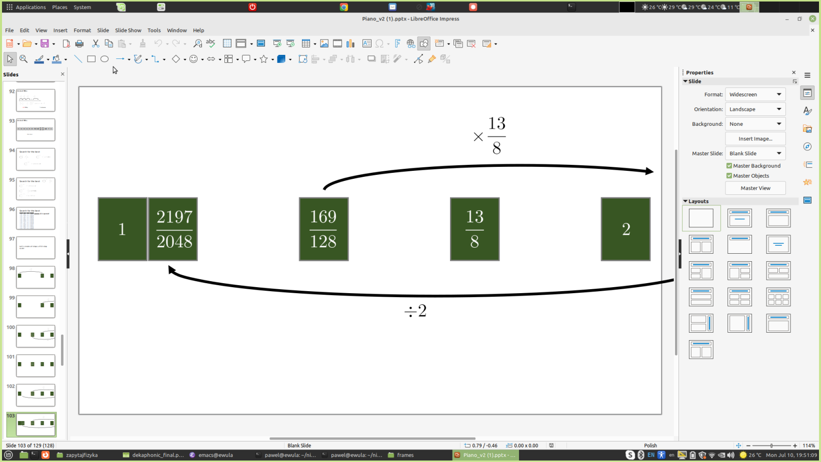
Ten scale Pythagorian tuning



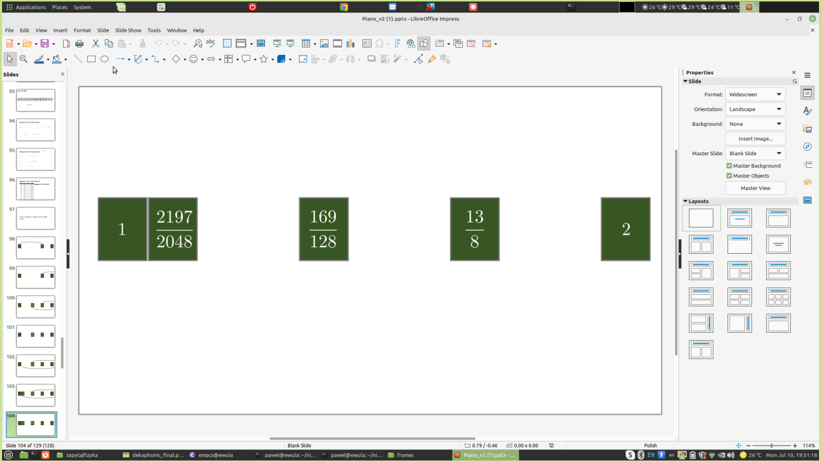
Ten scale Pythagorian tuning



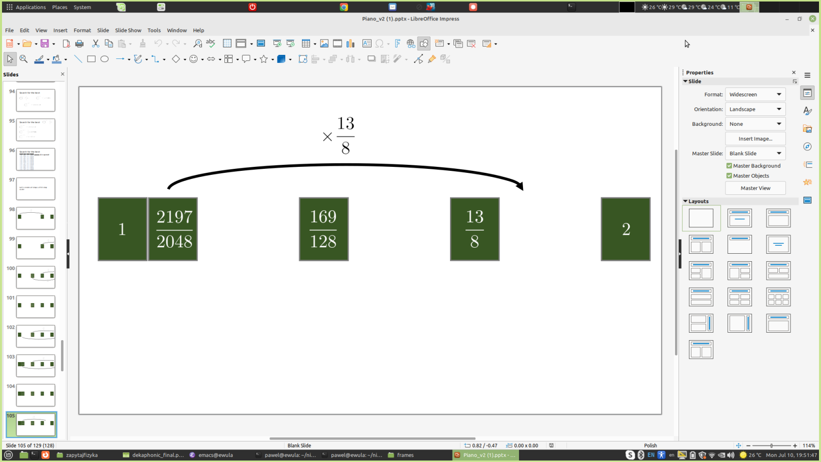
Ten scale Pythagorian tuning



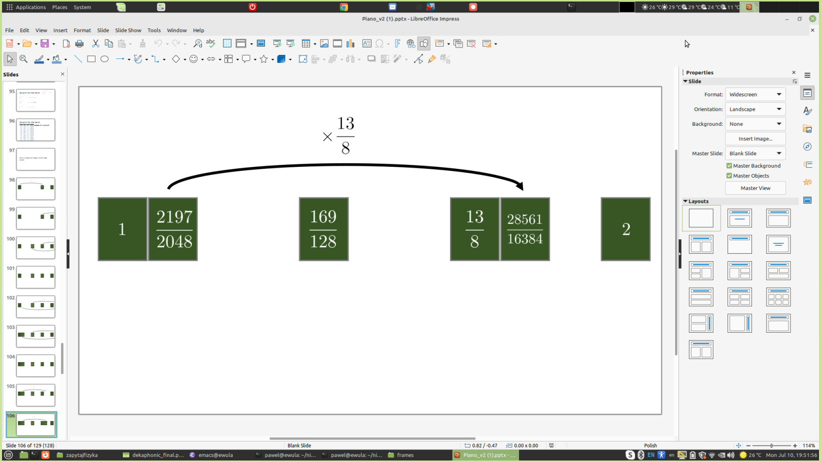
Ten scale Pythagorian tuning



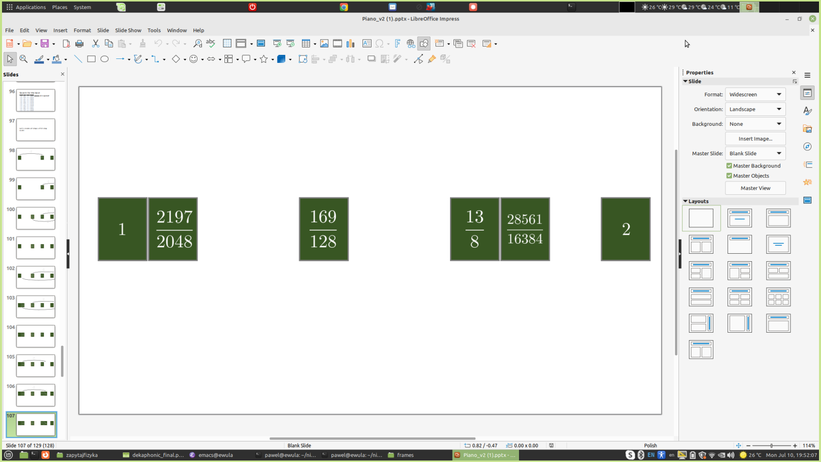
Ten scale Pythagorian tuning



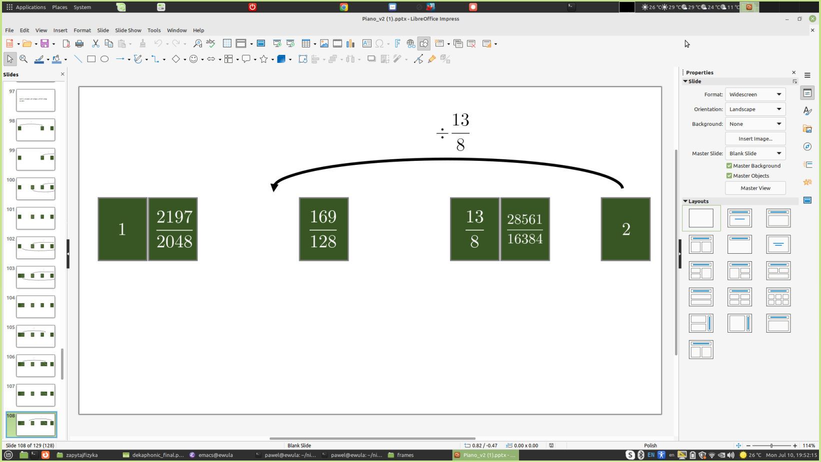
Ten scale Pythagorian tuning



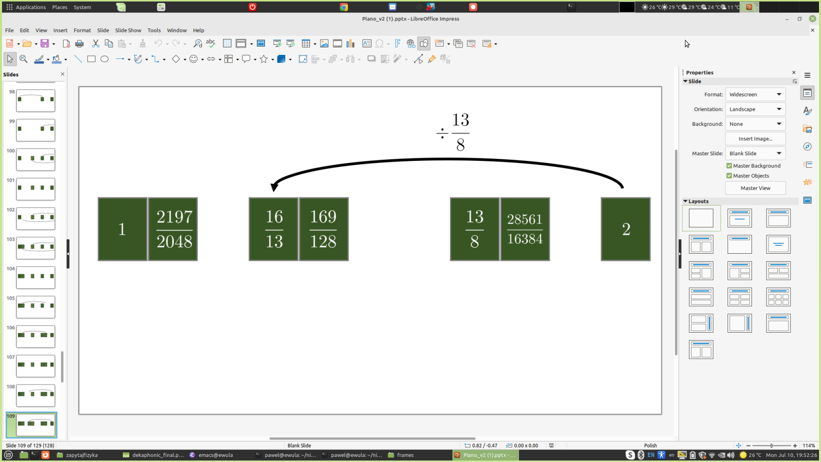
Ten scale Pythagorian tuning



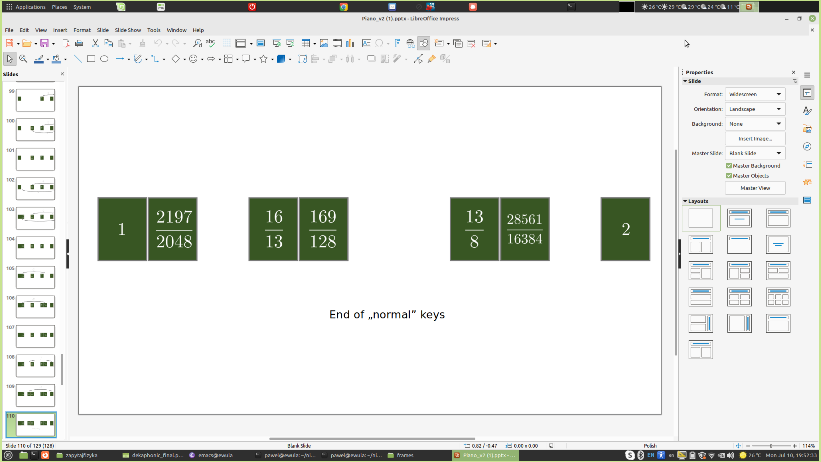
Ten scale Pythagorian tuning



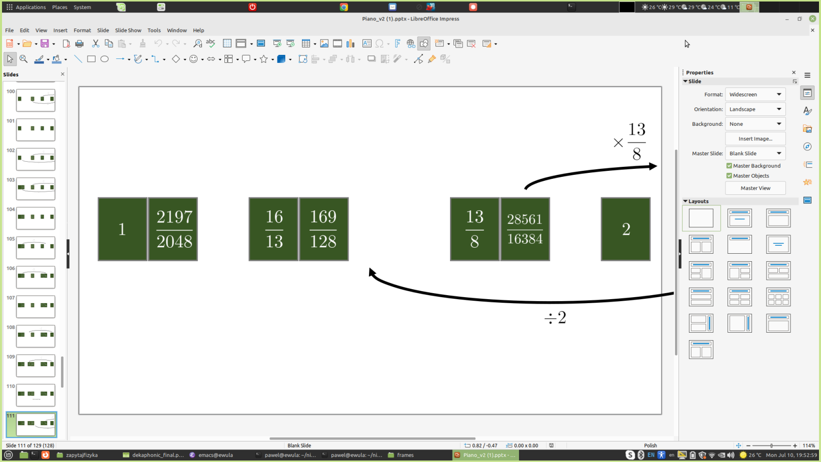
Ten scale Pythagorian tuning



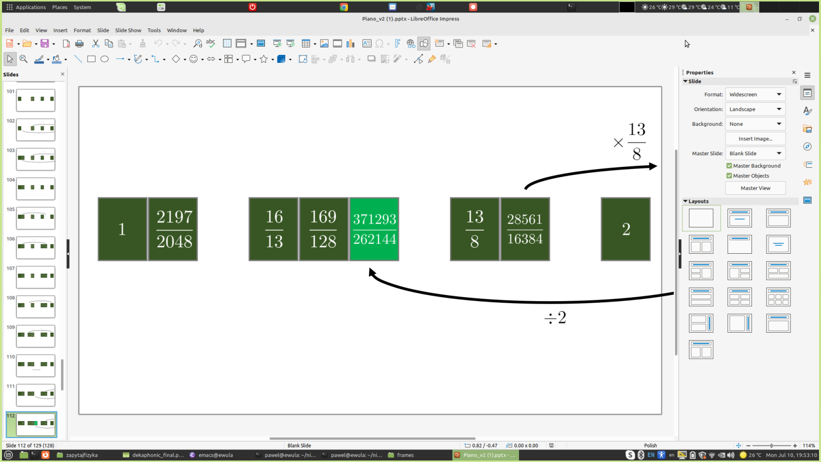
Ten scale Pythagorian tuning



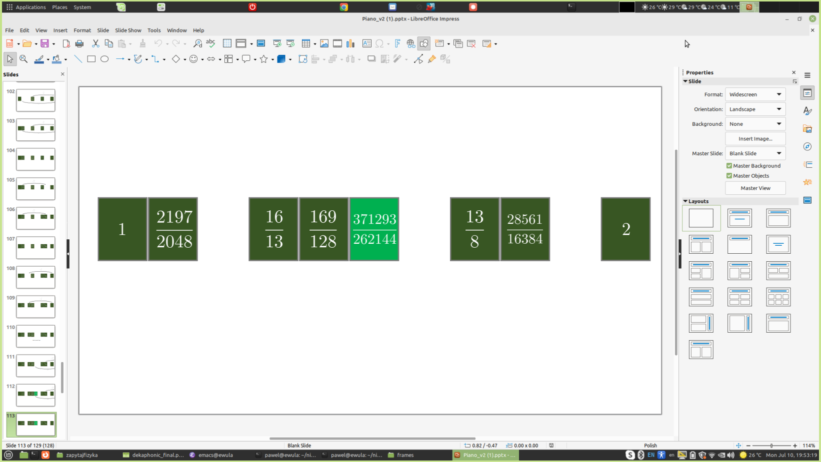
Ten scale Pythagorian tuning



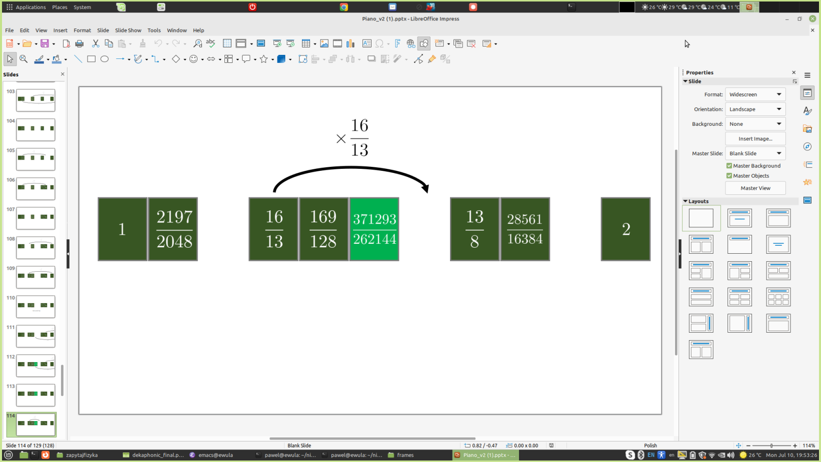
Ten scale Pythagorian tuning



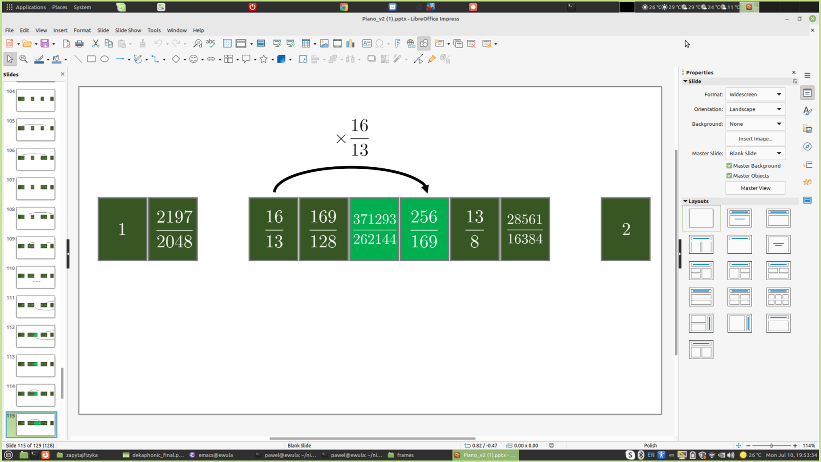
Ten scale Pythagorian tuning



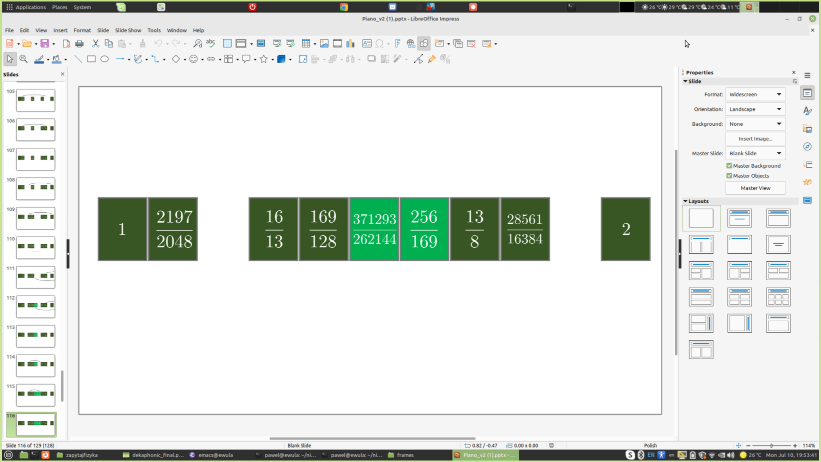
Ten scale Pythagorian tuning



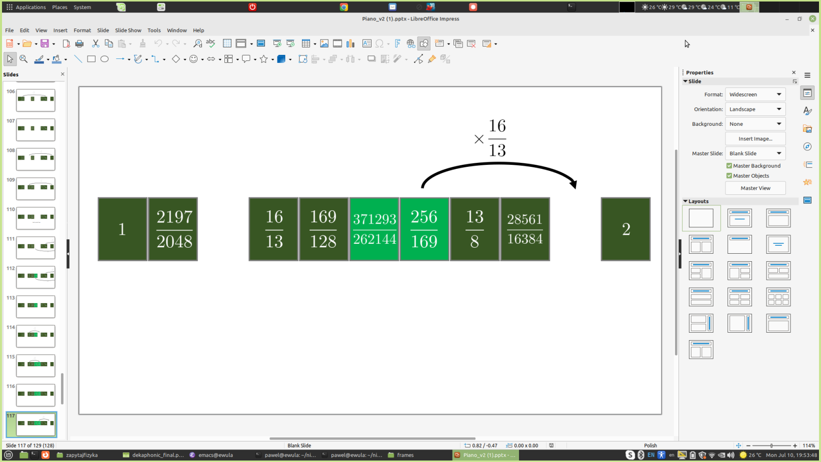
Ten scale Pythagorian tuning



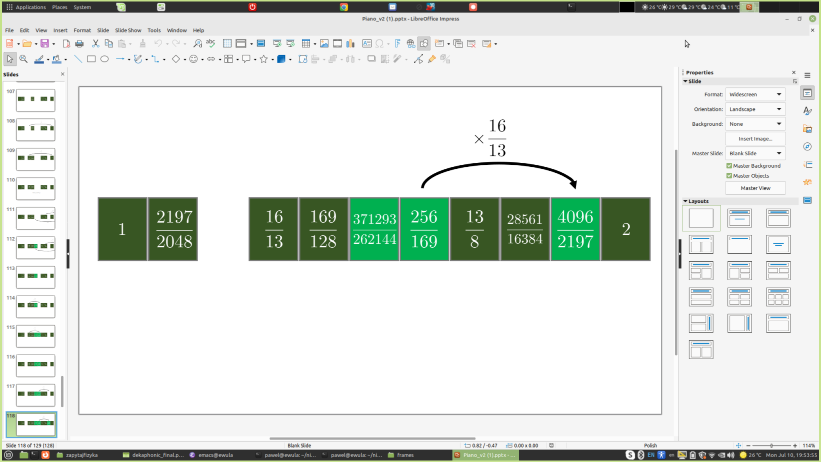
Ten scale Pythagorian tuning



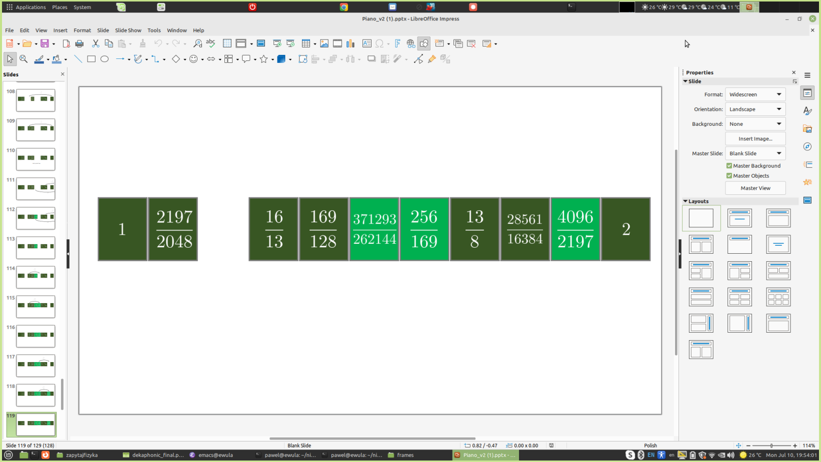
Ten scale Pythagorian tuning



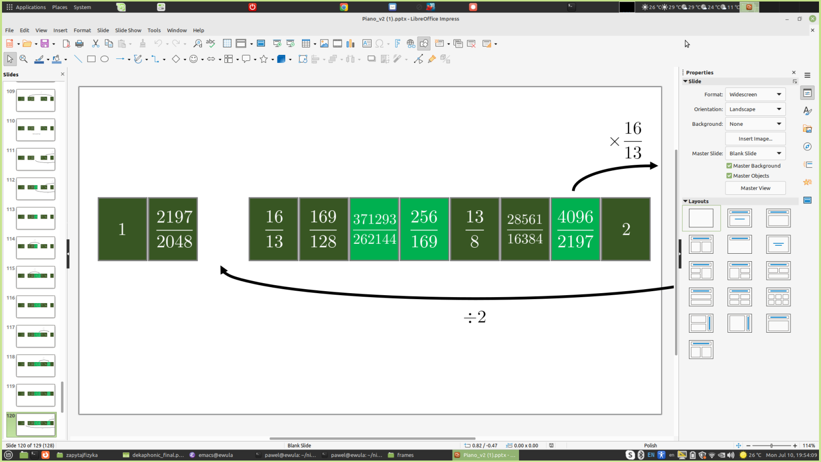
Ten scale Pythagorian tuning



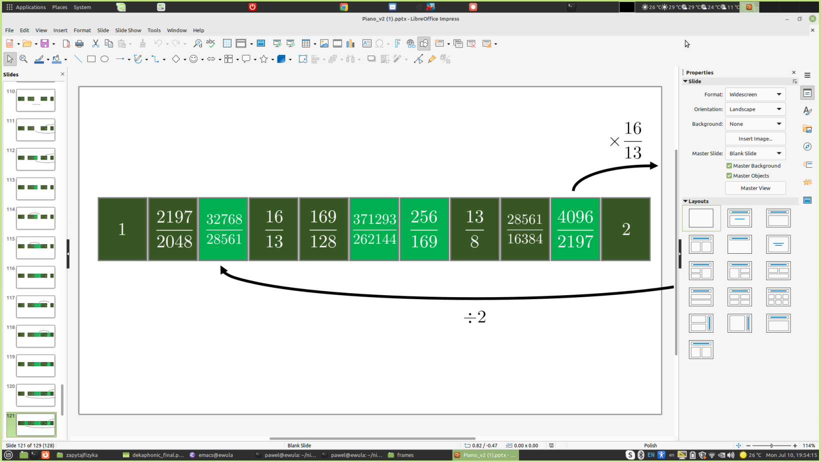
Ten scale Pythagorian tuning



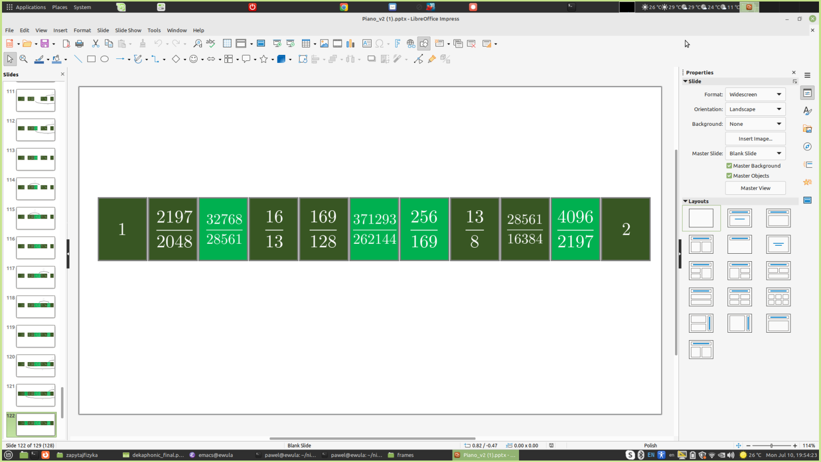
Ten scale Pythagorian tuning



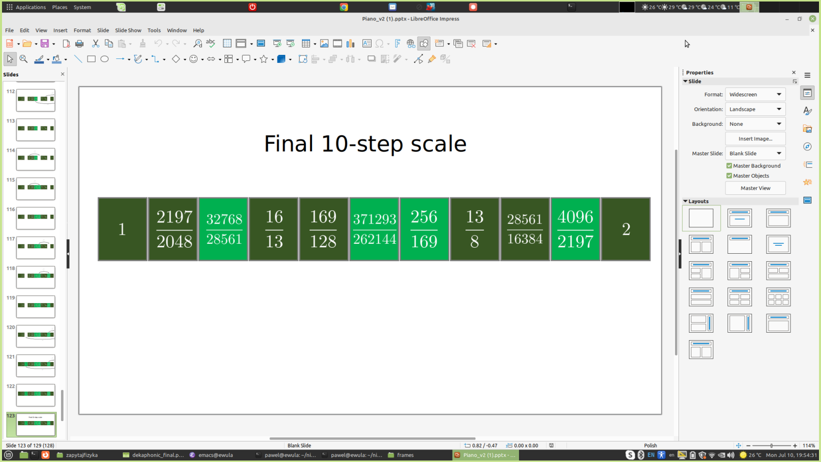
Ten scale Pythagorian tuning



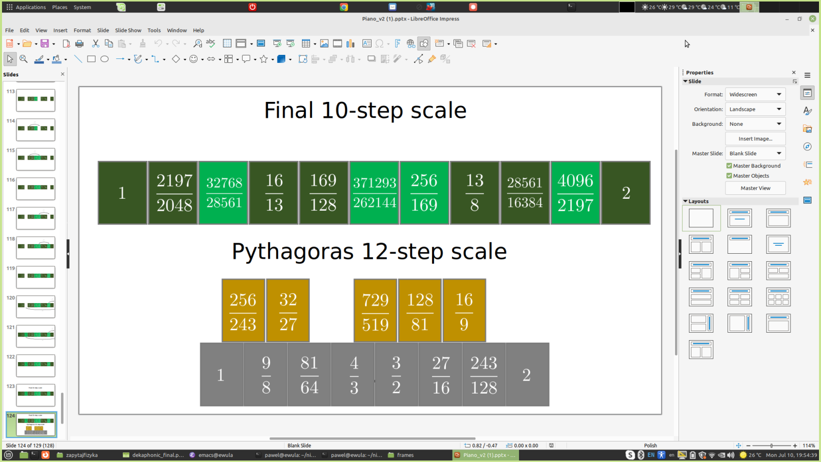
Ten scale Pythagorian tuning



Ten scale Pythagorian tuning



Ten scale Pythagorian tuning



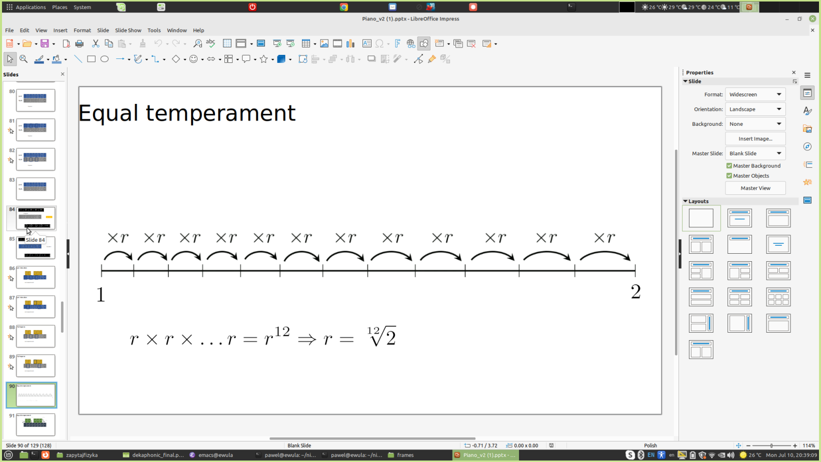
- Consider all possible musical systems with t keys in an octave.
- As far as the **problem of transposing melodies** from one key to another is concerned, the **best** among these systems is the t-system in which all the **adjascent keys are apart the same interval**.
- Let this interval be given by a number r.
- We start with our first key, in the first octave, which has frequency value 1.
- The second key has frequency value $1 \times r = r$, , the third key, $r \times r = r^2$, the fourth $r^2 \times r = r^3$, and so on, until the t^{th} key, which will have frequency value $r^{t-2} \times r = r^{t-1}$.
- Since the scale has t keys, the $(t+1)^{\text{th}}$ key **starts a new octave**, so that the value of this key is on one hand $r^{t-1} \times r = r^t$, but on the other hand is 2.

- We thus have $r^t = 2$, or r is the t^{th} **root of** r.
- Thus, the musical intervals in such equally distanced scale are equal to

$$r = (2)^{\frac{1}{t}}.$$

- A musical scale with t equal intervals $r = (2)^{\frac{1}{t}}$ is called **equally tempered**, or **equal temperament.**

Equal temperament for $t = 12$



- The **decaphonic piano of Leszek Mozdzer**, which physical implementation will be presented during the **concert at Nowa Miodowa Hall on Thursday, 13th July, at 19:00**, uses **equally tempered 10 scale musical system.**
- Mathematically the system has a remarkable property that the **differences in musical intervals** between **each of its ten keys** and the corresponding keys of **the Pythagorian** $f = 13$, $t = 10$, $k = 7$ **system**, are about **an order of magnitude smaller**, than the corresponding distances differences between keys of the **equally tempered 12 piano scale** and the keys of **its** $f = 3$, $t = 12$, $k = 7$ **Pytagorian counterpart.**